A New Solution for a Queueing Model of a Manufacturing Cell with Negative Customers under a Rotation Rule

Tien Van Do

Department of Telecommunications,
Budapest University of Technology and Economics,
H-1117, Magyar tudósok körútja 2., Budapest, Hungary.
Tel: +36 1 463 2070

Abstract

In this paper we consider a queueing model extension for a manufacturing cell composed of a machining center and several parallel downstream production stations under a rotation rule. A queueing model is extended with the arrival processes of negative customers to take into account failures of production stations, reorganization of works and disasters in the manufacturing cell. We present an exact solution for the steady state probabilities of the proposed queueing model. The solution does not require the approximation of the infinite sum. In addition, we provide an alternative way to compute the rate matrix for the matrix-geometric method as well.

Keywords: manufacturing cell; rotation rule; G-networks; negative customers; embedded Markov chain; matrix-geometric method;

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1 Introduction

The performance modeling and evaluation activities play an important role in the design and operation of manufacturing systems. Therefore, the search of new mathematical analysis methods for the performance evaluation of manufacturing systems has been an intensive research area [1–19].

Email address: `do@hit.bme.hu` (Tien Van Do).

One type of manufacturing systems is a manufacturing cell that is typically configured from a machining center and several parallel downstream production stations with buffer of infinite size [6]. Raw material from an ample supply is processed by the machining center (MC) for a number of different part-types. Then, the part is moved to downstream production stations for further processing after the completion of the part in the MC. A production sequence decides which type to produce upon the completion of the processing of the part in the machining center [6, 17].

A queueing model for a manufacturing cell composed of a machining center and several parallel downstream production stations under a rotation rule was introduced and analyzed in [6]. The matrix-geometric method [20] is applied by Chen [6] to calculate the stationary probabilities of the queueing model. The matrix-geometric method requires the computation of the sum to infinity in the iterative form of the matrix equation, however the infinite sum is not available in the closed form. Therefore, the infinite sum is approximated by the truncation [6].

In this paper, we propose a model extension for the manufacturing cell composed of a machining center and several parallel downstream production stations under a rotation rule with negative customers. The concept of G-networks with negative arrivals was first published by Gelenbe in 1989 [21–23]. Negative customers remove positive customers in the queue and have been used to model random neural networks, task termination in speculative parallelism, faulty components in manufacturing systems and server breakdowns and a reaction network of interacting molecules [21–29]. Queueing models with negative customers can account for burstiness and correlation, but in addition the negative customers, with an appropriate killing discipline, can represent additional behaviours such as breakdowns, killing signals, losses and load balancing [30–49].

The rest of this paper is organized as follows. In Section 2 an extension with two Poisson processes of negative customer arrivals is proposed to take into account extreme events. The first Poisson process is used to model the failure of the server in stations, while the second Poisson process is applied to capture catastrophic events in a specific station or the reorganization of processing parts in the manufacturing cell. A negative customer of the first Poisson process removes a part from a specific station in our model, while a negative customer of the second Poisson process kills all customers from the specific station. In Section 3 we present an exact method that does not require the truncation of an infinite sum. Finally, the paper is concluded in Section 4.

2 A Model with Negative Customers

Consider a queueing model with negative customers for a manufacturing cell that is composed of a machining center and m parallel downstream production stations with buffer of infinite size under a rotation rule.

The production sequence in a machining center consists of a repeated cycle that can be partitioned into scheduling lists for station j. The number of the scheduling lists for station j is equal to the occurrence number of type j parts in a repeated cycle. Each scheduling list starts from a node after type j node and ends with a type j node. The processing times of parts in station j follow the exponential distribution with rate μ_j.

To model extreme events we introduce two different Poisson processes with rate ϕ_j and φ_j at station j.

- The first Poisson process is used to model the failure of the server in station j. A negative customer of the first Poisson process removes a part being serviced from station j if any in our model.
- The second Poisson process is applied to capture catastrophic events in station j or the removal of parts due to the reorganization in the manufacturing cell. In our model a negative customer of the second Poisson process kills all the parts in station j.

Note that both the Poisson processes do not have impact to an empty station.

Example 1 *An example of such systems with four production stations is illustrated in Figure 1. Assume that a part of type 3, a part of type 1, a part of type 3, a part of type 4, a part of type 2, a part of type 3, a part of type 4, a part of type 1, a part of type 2, a part of type 3, a part of type 3, a part of type 4, a part of type 2 is repeatedly processed in the MC. Therefore, the production sequence consists of a repeated cycle in the MC:* $\{3 \to 1 \to 3 \to 4 \to 2 \to 3 \to 4 \to 1 \to 2 \to 3 \to 3 \to 4 \to 2\}$.

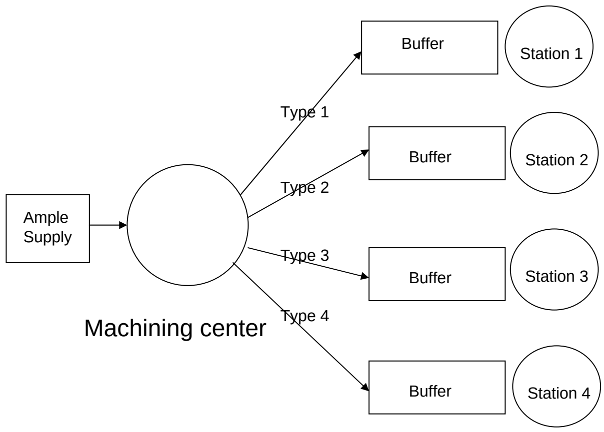

Fig. 1. Illustration for a system with four production stations

Some notations are introduced as follows:

- $L(j, i)$ is the sequence in the ith scheduling list of type j.

- $N(j)$ denotes the occurrence number of type j parts in a repeated cycle.
- $N(j,i,k)$ is the occurrence number of type k parts in the ith scheduling list of type j for $k \neq j$.

Remarks. Let us consider the second production station in Example 1. Then, there are three scheduling lists for the second station: $L(2,1) = \{3 \to 1 \to 3 \to 4 \to 2\}$, $L(2,2) = \{3 \to 4 \to 1 \to 2\}$ and $L(2,3) = \{3 \to 3 \to 4 \to 2\}$. Values for some notations are as follows:

- $N(1) = 2$, $N(2) = 3$, $N(3) = 5$, $N(4) = 3$,
- $N(2,1,1) = 1, N(2,1,3) = 2, N(2,1,4) = 1$,
- $N(2,2,1) = 1$, $N(2,2,3) = 1$, $N(2,2,4) = 1$,
- $N(2,3,1) = 0$, $N(2,3,3) = 2$, $N(2,3,4) = 1$.

Additional notations are as follows:

- Y_j denotes the processing time of type j parts in the MC. Random variable Y_j is exponentially distributed with mean $1/\lambda_j$.
- $G_{j,i}$ represents the interarrival time of parts at station j when the production sequence follows the ith scheduling list of type j.
- $G_{j,i}(t)$ is the cumulative distribution function of $G_{j,i}$, i.e., $G_{j,i}(t) = \Pr(G_{j,i} < t)$.
- $G^*_{j,i}(s)$ denotes Laplace-Stieltjes transform $G^*_{j,i}(s) = \int_0^\infty e^{-st} dG_{k,i}(t)$.
- $p_{k,i,j}$ is the probability that the number of type j parts completed in station j during $G_{k,i}$ is equal to k when the production sequence follows the ith scheduling list of type j.
- $\pi_{n,i,j}$ denotes the stationary probability that the number of parts remaining in station j is equal to n at the moment the MC completes a type j part and the production sequence follows the ith schedule list.

Let $\aleph_h$ denote the number of parts remaining in station j and $\daleth_h$ be the scheduling list seen by the hth arrival of a part at station j. Process $\{\aleph_h, \daleth_h, h \geq 1\}$ constitutes an embedded Markov chain on state space $\{(n,i) | n = 0, 1, \ldots, \infty; i = 1, 2, \ldots, N(j)\}$. Let $\pi_{n,i,j}$ denote the stationary probability as $\pi_{n,i,j} = \Pr_{h \to \infty}(\aleph_h = n, \daleth_h = i)$. We introduce vector $\boldsymbol{\alpha}_{n,j} = (\pi_{n,1,j}, \ldots, \pi_{n,N(j),j})$.

Following [6] we can write the following equations

$$G^*_{j,i}(s) = \frac{\lambda_j}{\lambda_j + s} \prod_{l=1,\neq j}^{m} \left(\frac{\lambda_l}{\lambda_l + s} \right)^{N(j,i,l)}. \tag{1}$$

It is not difficult to express the transition probabilities

$$p_{k,i,j} = \int_0^\infty \frac{((\mu_j + \phi_j)t)^k e^{-(\mu_j+\phi_j)t} e^{-\varphi_j t}}{k!} dG_{j,i}(t) = \frac{(-\mu_j - \phi_j)^k}{k!} G^{*(k)}_{j,i}(\mu_j + \phi_j + \varphi_j), \tag{2}$$

where $G_{j,i}^{*(k)}(s)$ is the order-k derivative of $G_{j,i}^{*}(s)$ with respect to s.

For $n \geq 0$, matrix $\mathbf{A}_{n,j}$ is introduced as follows:

$$\mathbf{A}_{n,j} = \begin{bmatrix} 0 & p_{n,2,j} & 0 & \dots 0 & 0 & 0 \\ 0 & 0 & p_{n,3,j} & \dots 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots & \vdots \\ 0 & 0 & & \dots 0 & 0 & p_{n,N(j),j} \\ p_{n,1,j} & 0 & & \dots 0 & 0 & 0 \end{bmatrix}, \ (n \geq 0).$$

Let us define $b_{n,i,j} = 1 - \sum_{l=0}^{n-1} p_{l,i,j}$ for $n = 1, 2, \dots$ and introduce vectors

$$\mathbf{B}_{n,j} = \begin{bmatrix} 0 & b_{n+1,2,j} & 0 & \dots 0 & 0 & 0 \\ 0 & 0 & b_{n+1,3,j} & \dots 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots & \vdots \\ 0 & 0 & & \dots 0 & 0 & b_{n+1,N(j),j} \\ b_{n+1,1,j} & 0 & & \dots 0 & 0 & 0 \end{bmatrix}, \ (n \geq 0).$$

Hence the transition probability matrix of the embedded Markov chain $\{\aleph_h, \daleth_h, h \geq 1\}$ can be written as follows

$$\begin{bmatrix} \mathbf{B}_{0,j} & \mathbf{A}_{0,j} & 0 & 0 & 0 & 0 & \dots \\ \mathbf{B}_{1,j} & \mathbf{A}_{1,j} & \mathbf{A}_{0,j} & 0 & 0 & 0 & \dots \\ \mathbf{B}_{2,j} & \mathbf{A}_{2,j} & \mathbf{A}_{1,j} & \mathbf{A}_{0,j} & 0 & 0 & \dots \\ \mathbf{B}_{3,j} & \mathbf{A}_{3,j} & \mathbf{A}_{2,j} & \mathbf{A}_{1,j} & \mathbf{A}_{0,j} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{3}$$

The stationary probabilities can be expressed according to the matrix-geometric method [20] as

$$\boldsymbol{\alpha}_{n,j} = \boldsymbol{\alpha}_{n,0} \mathbf{R}_j^n. \tag{4}$$

Hence the key step is to compute the rate matrix $\mathbf{R}_j$ that is the minimal nonnegative solution to the matrix equation

$$\mathbf{R}_j = \sum_{k=0}^{\infty} \mathbf{R}_j^k \mathbf{A}_{k,j}. \tag{5}$$

The rate matrix can be calculated by applying a numerically iterative procedure [20]. However, a numerically iterative procedure needs the approximation of the infinite sum in the right hand side of equation (5). Therefore, the infinite sum is approximated by $\sum_{k=0}^{K} \mathbf{R}_j^k \mathbf{A}_{k,j}$, where K is an appropriately large value such as the approach applied in [6]. We shall present a new method where the numerical approximation of the infinite sum is not needed in the subsequent section.

3 A Proposed Solution

The balance equations can be written as follows:

$$\boldsymbol{\alpha}_{0,j} = \sum_{n=0}^{\infty} \boldsymbol{\alpha}_{n,j} \mathbf{B}_{n,j}, \tag{6}$$

$$\boldsymbol{\alpha}_{n,j} = \sum_{k=-1}^{\infty} \boldsymbol{\alpha}_{n+k,j} \mathbf{A}_{k+1,j} \quad (n \geq 1). \tag{7}$$

In addition, the normalization equation is

$$\sum_{i=1}^{N(j)} \sum_{n=0}^{\infty} \pi_{n,i,j} = 1. \tag{8}$$

Let $\mathbf{Q}_j(x) = \mathbf{A}_{0,j} + (\mathbf{A}_{1,j} - \mathbf{I}_j)x + \sum_{k=2}^{\infty} \mathbf{A}_{k,j} x^k$ be defined as the characteristic matrix polynomial associated with equation (7), where $\mathbf{I}_j$ is the identity matrix of size $N(j) \times N(j)$. Note that the term of the characteristic matrix polynomial was introduced in [35,50–52] to obtain the stationary probabilities of Quasi-Birth-Death processes. In these works the degree of the characteristic matrix polynomial is two (see [50–52]) or finite [35].

Assume that $\mathbf{Q}_j(x)$ has d pairs of eigenvalue-eigenvectors $(x_{i,j}, \boldsymbol{\psi}_{i,j})$, thus satisfying the equations

$$\boldsymbol{\psi}_{i,j} \mathbf{Q}_j(x_{i,j}) = 0, \quad \text{for } i = 1, \ldots, d, \tag{9}$$

$$det[\mathbf{Q}_j(x_{i,j})] = 0, \quad \text{for } i = 1, \ldots, d. \tag{10}$$

For the k^{th} ($k = 1, \ldots, d$) eigenvalue-eigenvector pair, $(x_{k,j}, \boldsymbol{\psi}_{k,j})$, by substituting $\boldsymbol{\alpha}_{n,j} = \boldsymbol{\psi}_{k,j} x_{k,j}^n$ in the equations (7), this set of equations is satisfied. This means $\boldsymbol{\alpha}_{n,j} = \boldsymbol{\psi}_{k,j} x_{k,j}^n$ is a particular solution. As a consequence, the general solution for $\boldsymbol{\alpha}_{n,j}$ is a linear sum of all the factors ($\boldsymbol{\psi}_{k,j} x_{k,j}^n$). That is, we can write

$$\boldsymbol{\alpha}_{n,j} = \sum_{k=1}^{d} \beta_{k,j} x_{k,j}^n \boldsymbol{\psi}_{k,j} \ (n \geq 0), \tag{11}$$

where $\beta_{1,j}, \ldots, \beta_{d,j}$ are coefficients to be determined. In order to satisfy the normalization equation (8), only $|x_{i,j}| < 1$ should be considered. Without the loss of generality, let d be the number of these eigenvalues $x_{i,j}$. The number of coefficients $\beta_{1,j}, \ldots, \beta_{d,j}$ to be determined is d. We can utilize equation (6) and the normalization equation to compute the coefficients. Thus, the number of linear equations is $N(j)+1$. However, only $N(j)$ linear equations are linearly independent. Therefore, the linear equations have the unique solution if and only if $N(j) = d$. This means, the exact number of eigenvalues inside the unit circle (i.e., $|x_{i,j}| < 1$) is $N(j)$.

Therefore, we have to determine the eigenvalues, the eigenvectors and the coefficients for the stationary probabilities.

3.1 Computation of Eigenvalues and Eigenvectors

Lemma 1 *For the stationary probability $p_{k,i,j}$ that the number of parts remaining in station j is equal to n at the moment the MC completes a type j part and the production sequence follows the ith schedule list,*

$$\sum_{k=0}^{\infty} x^k p_{k,i,j} = G_{j,i}^*((\mu_j + \phi_j)(1 - x) + \varphi_j). \tag{12}$$

Proof. This is clearly

$$\begin{aligned}
\sum_{k=0}^{\infty} x^k p_{k,i,j} &= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{(x(\mu_j + \phi_j)t)^k e^{-(\mu_j+\phi_j)t} e^{-\varphi_j t}}{k!} dG_{j,i}(t) \\
&= \int_0^{\infty} \left(\sum_{k=0}^{\infty} \frac{(x(\mu_j + \phi_j)t)^k}{k!} \right) e^{-(\mu_j+\phi_j)t} e^{-\varphi_j t} dG_{j,i}(t) \\
&= \int_0^{\infty} e^{x(\mu_j+\phi_j)t} e^{-(\mu_j+\phi_j)t} e^{-\varphi_j t} dG_{j,i}(t) \\
&= G_{j,i}^*((\mu_j + \phi_j)(1 - x) + \varphi_j). \qquad \square
\end{aligned} \tag{13}$$

By Lemma 1 we obtain the closed form for the characteristic matrix polynomial as

$$\mathbf{Q}_j(x) = \mathbf{A}_{0,j} + (\mathbf{A}_{1,j} - \mathbf{I}_j)x + \sum_{k=2}^{\infty} \mathbf{A}_{k,j}x^k = -\mathbf{I}_j x + \sum_{n=0}^{\infty} \mathbf{A}_{n,j}x^n,$$

$$\mathbf{Q}_j(x) = \begin{bmatrix} -x & \sum_{n=0}^{\infty} p_{n,2,j}x^n & 0 & \dots 0 & 0 & 0 \\ 0 & -x & \sum_{n=0}^{\infty} p_{n,3,j}x^n & \dots 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots \\ 0 & 0 & & \dots 0 & -x & \sum_{n=0}^{\infty} p_{n,N(j),j}x^n \\ \sum_{n=0}^{\infty} p_{n,1,j}x^n & 0 & & \dots 0 & 0 & -x \end{bmatrix} =$$

$$\begin{bmatrix} -x & G^*_{j,2}(\omega_j) & 0 & \dots 0 & 0 & 0 \\ 0 & -x & G^*_{j,3}(\omega_j) & \dots 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots \\ 0 & 0 & & \dots 0 & -x & G^*_{j,N(j)}(\omega_j) \\ G^*_{j,1}(\omega_j) & 0 & & \dots 0 & 0 & -x \end{bmatrix},$$

where $\omega_j = (\mu_j + \phi_j)(1 - x) + \varphi_j$. The eigenvalues $x_{i,j}$'s ($i = 1, \dots, N(j)$) inside the unit circle can be easily obtained because of the closed form of the characteristic matrix polynomial $\mathbf{Q}_j(x)$.

Because of the special structure of the characteristic matrix polynomial $\mathbf{Q}_j(x)$ we obtain

$$det[\mathbf{Q}_j(x)] = (-x)^{N(j)} - (-1)^{N(j)} \prod_{i=1}^{N(j)} G^*_{j,i}(\omega_j). \tag{14}$$

Substituting (1) into equation (14), we get

$$
\begin{aligned}
det[\mathbf{Q}_j(x)] &= (-x)^{N(j)} - (-1)^{N(j)} \prod_{i=1}^{N(j)} \frac{\lambda_j}{\lambda_j+\omega_j} \prod_{k=1,\neq j}^{m} \left(\frac{\lambda_k}{\lambda_k+\omega_j}\right)^{N(j,i,k)} \\
&= (-x)^{N(j)} - (-1)^{N(j)} \left(\frac{\lambda_j}{\lambda_j+\omega_j}\right)^{N(j)} \prod_{i=1}^{N(j)} \prod_{k=1,\neq j}^{m} \left(\frac{\lambda_k}{\lambda_k+\omega_j}\right)^{N(j,i,k)} \\
&= (-x)^{N(j)} - (-1)^{N(j)} \left(\frac{\lambda_j}{\lambda_j+\omega_j}\right)^{N(j)} \prod_{k=1,\neq j}^{m} \left(\frac{\lambda_k}{\lambda_k+\omega_j}\right)^{\sum_{i=1}^{N(j)} N(j,i,k)} .
\end{aligned}
\tag{15}
$$

Since there are various technique to find the roots of polynomials, we can easily find the roots of $det[\mathbf{Q}_j(x)] = 0$ and therefore the eigenvalues ($x_{i,j}$'s for $i = 1, \ldots, N(j)$ inside the unit circle) of the characteristic matrix polynomial $\mathbf{Q}_j(x)$ as well.

The eigenvectors are calculated as follows. Let $\boldsymbol{\psi}_{i,j} = [\psi_{i,j,1}, \psi_{i,j,2}, \ldots, \psi_{i,j,N(j)}]$ be an eigenvector for the eigenvalue $x_{i,j}$ ($i = 1, 2, \ldots, N(j)$). Expanding equation (9), we get the recursive relations $\psi_{i,j,l+1} = \psi_{i,j,l} G^*_{j,l+1}(\omega_j)/x$ between $\psi_{i,j,l}$ and $\psi_{i,j,l+1}$ for $l = 1, \ldots, N(j) - 1$. An eigenvector remains as the eigenvector corresponding to the same eigenvalue when it is multipled by a scalar. Using this property, we can determine $\boldsymbol{\psi}_{i,j}$ by setting $\psi_{i,j,1} = 1$ and applying the above recursive relations to compute $\psi_{i,j,l}$ for $l = 2, \ldots, N(j)$.

3.2 Computation of Coefficients

Lemma 2 *The right hand side of balance equation (6) can be expressed as follows*

$$
\sum_{n=0}^{\infty} \boldsymbol{\alpha}_{n,j} \mathbf{B}_{n,j} = \sum_{k=1}^{N(j)} \frac{\beta_{k,j}}{1-x_{k,j}} \boldsymbol{\psi}_{k,j} \mathbf{E}_j - \sum_{k=1}^{N(j)} \frac{\beta_{k,j}}{1-x_{k,j}} \boldsymbol{\psi}_{k,j} (\mathbf{Q}_j(x_{k,j}) + x_{k,j} \mathbf{I}_j), \tag{16}
$$

where $\mathbf{E}_j$ is a matrix of size $N(j) \times N(j)$ as

$$\mathbf{E}_j = \begin{bmatrix} 0\ 1\ 0 \ldots 0\ 0\ 0 \\ 0\ 0\ 1 \ldots 0\ 0\ 0 \\ \vdots\ \vdots\ \vdots\ \ \vdots\ \ \vdots\ \vdots\ \vdots \\ 0\ 0 \quad \ldots 0\ 0\ 1 \\ 1\ 0 \quad \ldots 0\ 0\ 0 \end{bmatrix}.$$

Proof. Since $b_{n,i,j} = 1 - \sum_{l=0}^{n-1} p_{l,i,j}$ for $n = 1, 2, \ldots$ by definition, $\mathbf{B}_{n,j} + \sum_{i=0}^{n} \mathbf{A}_{i,j} = \mathbf{E}_j$ holds. Therefore, we obtain

$$\sum_{n=0}^{\infty} \boldsymbol{\alpha}_{n,j} \left(\mathbf{B}_{n,j} + \sum_{i=0}^{n} \mathbf{A}_{i,j} \right) = \sum_{n=0}^{\infty} \boldsymbol{\alpha}_{n,j} \mathbf{E}_j$$
$$= \sum_{n=0}^{\infty} \sum_{k=1}^{N(j)} \beta_{k,j} x_{k,j}^n \boldsymbol{\psi}_{k,j} \mathbf{E}_j = \sum_{k=1}^{N(j)} \frac{\beta_{k,j}}{1 - x_{k,j}} \boldsymbol{\psi}_{k,j} \mathbf{E}_j. \tag{17}$$

We can derive the following relation

$$\sum_{n=0}^{\infty} \boldsymbol{\alpha}_{n,j} \sum_{i=0}^{n} \mathbf{A}_{i,j} = \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \boldsymbol{\alpha}_{n,j} \mathbf{A}_{i,j} = \sum_{k=1}^{N(j)} \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \beta_{k,j} x_{k,j}^n \boldsymbol{\psi}_{k,j} \mathbf{A}_{i,j}$$
$$= \sum_{k=1}^{N(j)} \frac{\beta_{k,j}}{1 - x_{k,j}} \boldsymbol{\psi}_{k,j} \sum_{i=0}^{\infty} \mathbf{A}_{i,j} x_{k,j}^i = \sum_{k=1}^{N(j)} \frac{\beta_{k,j}}{1 - x_{k,j}} \boldsymbol{\psi}_{k,j} (\mathbf{Q}_j(x_{k,j}) + x_{k,j} \mathbf{I}_j). \tag{18}$$

Using (17) and (18) this yields equation (16). □

By Lemma 2, balance equation (6) can be rewritten as follows

$$\boldsymbol{\alpha}_{0,j} = \sum_{k=1}^{N(j)} \frac{\beta_{k,j}}{1 - x_{k,j}} \boldsymbol{\psi}_{k,j} \mathbf{E}_j - \sum_{k=1}^{N(j)} \frac{\beta_{k,j}}{1 - x_{k,j}} \boldsymbol{\psi}_{k,j} (\mathbf{Q}_j(x_{k,j}) + x_{k,j} \mathbf{I}_j). \tag{19}$$

From (11), the normalization equation can be rewritten as follows

$$\sum_{n=0}^{\infty} \sum_{k=1}^{N(j)} \beta_{k,j} x_{k,j}^n \boldsymbol{\psi}_{k,j} \boldsymbol{e} = \sum_{k=1}^{N(j)} \frac{\beta_{k,j}}{1 - x_{k,j}} \boldsymbol{\psi}_{k,j} \boldsymbol{e}_j = 1, \tag{20}$$

where $\boldsymbol{e}_j$ is a vector of size $N(j)$ with all elements equal to 1.

To compute the coefficients $\beta_{1,j}, \ldots, \beta_{N(j),j}$, we have to utilize equations (19) and (20), which form the system of $N(j)$ independent linear equations. Note that no direct computation of the infinite sums is needed in equations (19) and (20).

3.3 Remarks

From equations (4) and (11), the rate matrix $\mathbf{R}_j$ can be obtained from the eigenvalues and eigenvectors of $Q_j(x)$ using simple algebraic work as follows

$$\mathbf{R}_j = \Psi_j^{-1} \cdot diag(x_{1,j}, \ldots, x_{N(j),j}) \cdot \Psi_j, \tag{21}$$

where

$$\Psi_j = \begin{bmatrix} \psi_{1,j,1} & \psi_{1,j,2} & \psi_{1,j,3} & \cdots & \psi_{1,j,N(j)} \\ \psi_{2,j,1} & \psi_{2,j,2} & \psi_{2,j,3} & \cdots & \psi_{2,j,N(j)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{N(j),j,1} & \psi_{N(j),j,2} & \psi_{N(j),j,3} & \cdots & \psi_{N(j),j,N(j)} \end{bmatrix}. \tag{22}$$

Then equations $\boldsymbol{\alpha}_{0,j} = \boldsymbol{\alpha}_{0,j}(\mathbf{I}_j - \mathbf{R}_j)^{-1}(\mathbf{E}_j^{-1} - \mathbf{R}_j)$ and $\boldsymbol{\alpha}_{0,j}(\mathbf{I}_j - \mathbf{R}_j)^{-1}\mathbf{e}_j = 1$ can be used to compute $\boldsymbol{\alpha}_{0,j}$ and the stationary probabilities. However, this way for the stationary probabilities is more computationally extensive than solving for the coefficients $\beta_{1,j}, \ldots, \beta_{N(j),j}$ because two matrix inversions (Ψ_j^{-1} and $(\mathbf{I}_j - \mathbf{R}_j)^{-1}$) are needed in addition.

4 Conclusion

In this paper, we have extended a model for the manufacturing cell composed of a machining center and several parallel downstream production stations under a rotation rule. The arrival processes of negative customers are used to model failures and disasters.

We have provided an exact solution for the proposed queueing model. The solution has an advantage over existing method because it does not require the approximation of the infinite sum. Moreover, an alternative procedure is presented to compute the rate matrix for the matrix-geometric method.

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Vitae

Tien Van Do attended the Bonnal Ngo Quyen high-school in Hai Phong, Vietnam. He received the M.Sc. and Ph.D. degrees in telecommunications engineering from the Technical University of Budapest, Hungary, in 1991 and 1996, respectively. At present he is an associate professor in the Department of Telecommunications of the Budapest University of Technology and Economics, and a leader of Communications Network Technology and Internetworking Laboratory. He has participated in the COPERNICUS-ATMIN 1463, the FP4 ACTS AC310

ELISA, FP5 HELINET, FP6 CAPANINA projects funded by EC and leads various projects on network planning, software implementations (ATM & IP network planning software, GGSN tester, program for IMS performance testing, VoIP measurement,etc.), test and performance evaluation with industry partners. His research interests are queuing theory, telecommunication networks, performance evaluation and planning of telecommunication networks.