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Numerical Distortion in Single-Tone DDS

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***Abstract* – Spurious performance of direct digital synthesis (DDS) is partly caused by two quantization operations in its numerical (i.e. digital) part. These errors are deterministic and periodic in the time domain, therefore they appear as line spectra (undesired components: spurs) in the frequency domain. Hence it is quite natural to analyze the effects by DFT (discrete Fourier transform). The amplitude quantization (AQ), being present permanently, causes harmonically related spurs, while phase truncation (PT) produces spurs around the output frequency by phase modulation. However, as a consequence of DDS sampling process, spurs would be folding back into the DDS bandwidth (first Nyquist zone) and possibly overlapping. A simple procedure is presented for evaluating location and level of spurs, which are due to numerical distortion in a standard DDS system. Examples using an interactive math tool are available online.**

***Keywords* – DDS, NCO, amplitude quantization, phase truncation, spectral purity, simulation.**

I. INTRODUCTION

Direct digital frequency synthesis (DDFS or simply DDS) generates real-life waveforms of repetitive nature by using digital data and mixed/analog signal processing blocks. The open-loop DDS is used especially for precise, fast frequency and phase tunable output. Solutions can be implemented in LSI (large-scale integration) and they play an ever-increasing role in digital waveform and agile clock generation and modulation.

The high-level architecture of a generic DDS system can be viewed as a simple assembly containing three parts (Fig. 1). A fixed rate clock is the time *reference* part. Numerically controlled oscillator (NCO), the *digital* part, consists of an overflowing phase accumulator (ACC, register length *r*-bit) and a lookup table (LUT, memory address length *m*-bit, data width *n*-bit). Here we model all techniques of waveform mapping as a simple lookup table operation. The *mixed/analog* part reconstructs the analog wave with a digital to analog converter (DAC) and anti-imaging filter (AIF).

NCO-based DDS is a point(memory location)-skipping technique and runs at a constant update(clock)-rate. There are several applications that do not convert the numerical samples into an analog signal, as it is the case in many digital communication systems (e.g. digital radios and modems, software-defined radios, digital down/up converters etc.).

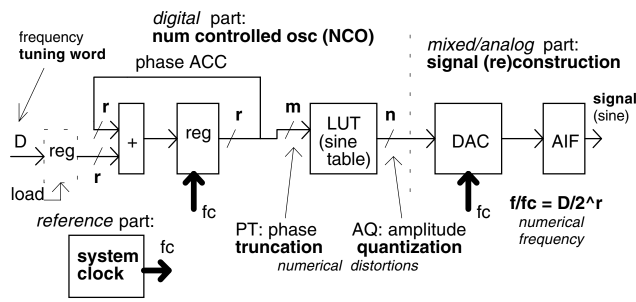

Fig. 1. The standard DDS structure.

One of the most important specifications to the synthesis is that of spectral purity. Spurious performance of DDS is partly caused by two quantization operations in its numerical (i.e. digital) part. The amplitude quantization causes harmonically related spurs, while phase truncation produces spurs around the output frequency by phase modulation. These numerical distortions (algorithmic nonlinearities) due to finite-wordlength effects present a major contribution to system complexity. (*Note*: the actual realization of DAC is another dominant source of spectral impurity, but here we concentrate to spurs origin from numerical part only.) Knowing the location and level of spurs is a good starting point for the NCO design, selection or length customizing (as in FPGA: field programmable gate array) for specific application.

Analytical results (particularly for amplitude quantization) are rather complicated, not necessarily practically oriented and because of spurs vary rather irregularly with tuning, interpreting the parameter dependence of spurs is not easy. This problem can be overcome by interactive computer simulation knowing the fundamental spur structures. The paper presents a simple procedure for evaluating numerical spurs in a standard single-tone (sinewave) DDS generator. The method can be extended to multi-tone or dithered DDS, as well. Examples using an *interactive* math tool (Mathcad worksheets) are available online.

II. NUMERICALLY CONTROLLED OSCILLATOR

First recollect some familiar results regarding DDS numerical frequency and numerical period of samples. Consider a continuous-time, normalized amplitude sinusoid with f analog frequency and a fixed zero initial phase: $x(t) = 1 \cdot \sin(2\pi f t)$, $|x(t)| \leq 1$. DDS builds up (reconstructs) the waveform from its numerical samples. Sampled at $f_c = 1/\Delta t$ rate, the discrete-time version of sinusoid: $x_i = x(i \cdot \Delta t) = 1 \cdot \sin(2\pi (f/f_c) \cdot i)$, where i: time index. The sample sequence x_i is P-periodic only, if the numerical frequency is rational: $f/f_c = L/P$ (some L relative prime to P). This is the case of so-called coherent sampling. The finite memory requirement of the DDS imposes the periodicity $x_i = x_{i+P}$.

The rate at which the r-bit phase accumulator overflows in NCO is controlled by the D frequency *tuning word*. Accumulator overflow (mod 2^r) corresponds to a mod 2π operation: $D/2^r = \Delta\theta / 2\pi$, so the output frequency (the rate of phase change): $f = (1/2\pi) \cdot (\Delta\theta / \Delta t) = (f_c / 2^r) \cdot D$. Hence the *numerical frequency* of DDS

$$f/f_c = D/2^r = L/P < 1/2, \text{ some } L \text{ (prime to } P\text{)},$$

and $P = 2^r / \gcd(D, 2^r) \leq 2^r$ is the *numerical period* of sample sequence. *Note*: **gcd** = greatest common divisor, and P is a power of 2.

A simple algorithm describes the operation of the NCO:

The input to NCO is the *integer D* tuning word. The actual content of an r-bit phase accumulator is $A_i = \mathrm{mod}(D \cdot i, 2^r)$. *Note*: **mod**$(a, b)$ = remainder of a when divided by b. Only high m-bit will index the LUT: $I_i = \mathrm{trunc}(A_i / 2^{r-m})$. There is one cycle of sine waveform in LUT, so a high precision table output is $T_i = 1 \cdot \sin((2\pi/N) \cdot I_i)$ and $N = 2^m$, but actually the NCO output with n-bit precision: $Q_i = \Delta \cdot \mathrm{round}(T_i / \Delta)$, where $\Delta = 2/2^n$.

Since the output numerical sequence Q_i is P-periodic, a P-point FFT transform of Q_i results in the *exact* output spectrum of NCO. Frequency indices greather than $P/2$ are redundant for real signals.

Varying L, prime to a given P, is equivalent with uniform(U)-*permutation* of samples within a P-block waveform (i.e. temporal reversible rearrangement of sample position i into position j, with $j = D \cdot i$ (mod P), $i = 0,1,2 \ldots P-1$), as showed by Kak and Jayant (1977). U-permutations produce an uniform shuffling in the DFT spectrum, as well. This means that only the position of the lines change but not its level, and for example the SNR (signal to noise (i.e. spur) ratio for full DDS bandwidth) is independent of L. The inverse mapping is another U-permutation.

Some remarks: (1) Simply change $I_i \rightarrow T_i$ mapping for smaller amplitude, nonzero phase or just another arbitrary (ARB) waveform or special sample-generation *simulation*. (2) One can use dither before phase or amplitude quantization if any. (3) Other type of quantization can be use for I_i or Q_i.

So far it seems that the procedure is free from major difficulties. However, for example, if D is odd then $P = 2^r$. Since r can be as high as 32 (or 48) for fine frequency resolution, extremely long periods can occur. In these cases practically some short-term FFT (with reasonable window against FFT artifacts), i.e. a global viewpoint can be used. But knowing the fine structure of spur locations, a local viewpoint: zoom DFT on interested area (on only suspect frequencies) also can be applied.

III. NUMERICAL DISTORTION

Amplitude quantization (AQ), being present permanently, and phase truncation (PT) are the two separate mechanisms which lead to the occurrence of spurs in NCO. These spurs vary rather irregularly with tuning word (D) and parameters (r, m, n), so revealing the fundamental structure of spur locations helps in interpreting the spectrum, as well.

A. *AQ-spurs: overlapped harmonic structure*

An n-bit amplitude quantization (AQ) is a strong deterministic nonlinearity, hence its rigorous analysis is quite complicated (if possible). But in some cases the behavior of AQ-error could be exactly quantified. Clavier, et al. (1947) provided an exact analysis when a uniform quantizer is driven by a continuous-time sinusoid. (Theoretical results were rediscovered later several times, sometimes without reference to the original paper.) Rounding operation, i.e. minimum distance (nearest neighbor) mapping has odd symmetry and generates odd harmonics only, the level of which oscillates violently and its dependence on the signal amplitude and resolution is very complicated.

Moreover, because of sampling, some harmonics can fold back into the desired band and possibly overlap depending on value of numerical frequency. Overlapped spectral lines may contribute constructively or destructively to the level of the resulting component depending on their actual phase (and amplitude) values. Therefore the practical way to *deterministic* analysis of the irregular shape of AQ-spur levels is directly given by an FFT transform of the NCO output.

Because of odd harmonics only, the number of AQ-spurs is $(P/4)-1$.

Aliasing phenomenon exposures the locations of AQ-spur (overlapped harmonics). Lines in the odd Nyquist zones map directly into the baseband (1^{st} Nyquist zone, single sided), while components in the even zones map in a mirrored fashion with phase inversion

$$z_h = \mathrm{mod}(h \cdot D, S), \quad S = 2^r \text{ and } h = 1, 3, 5 \ldots (P/2)-1$$

$$sa_h = \mathrm{if}\ (z_h < (S/2), z_h, (S - z_h)) .$$

Note: the expression **if**(*cond*, *t*, *f*) equals to *t* if *cond* holds and equals to *f* otherwise. Here $h = 1$ is the signal component. The analog frequency of AQ-spurs is $fa_h = (sa_h / S) \cdot f_c$.

Some consequences: (1) There is no numerical distortion especially if $P = 4$, because samples match exactly the quantization levels. At this $f / f_c = 1/4$ numerical frequency only the *DAC nonlinearities* cause distortion in DDS. (2) The worst case, a highly massive overlapping occurs at $P = 8$ (i.e. if $f / f_c = 1/8$ or $3/8$) when energy of spurs apparently "concentrates" on one AQ-spur. A simple estimate of bound on maximum AQ-spur to signal (carrier) ratio is SpSR $\approx (-6 \cdot n + 3)$ dBc. (3) The other limiting case occurs if P is long, then a "sea" of finely spaced AQ-spurs appears (or alternatively one can detect a background "noise floor").

B. PT-spurs: overlapped modulation structure

Truncating an r-bit phase accumulator output to high m-bit, modifies accordingly the effect of D tuning word on instantaneous phase that leads to phase truncation (PT) - error, but do not modify the average numerical frequency. Dropping the lower bits permits extremely fine-tuning with reasonable LUT size at the expense of spectral purity.

Let d be the nominal frequency tuning word (i.e. the nominal address change of LUT) and let us write it in *mixed fraction* form:

$$d = D/2^{r-m} = V + (L/E),$$

here $V = \mathrm{trunc}(d)$ is the integer part and $(L/E) = \mathrm{mod}(d,1)$ is the fractional part, where $E = 2^{r-m}/\gcd(D, 2^{r-m}) \leq 2^{r-m}$ can be identified as the *numerical period of phase error* (as we see next).

From this form, it is obvious that the actual address change is V or V+1 (if d is not integer), i.e. a "phase hiccap" occurs. Or viewing it another way: if $\mathrm{mod}(d,1) \neq 0$ then nonuniform sampling occurs and for this reason distortion is presented. Notice that if d is integer (i.e. if $E = 1$), then there are no PT-spurs.

The source of the PT-spurs is a fractional part of d. For the calculation of spur locations consider a discrete-time sinusoid

$$x_i = \sin(2\pi \frac{D}{2^r} i) = \sin(2\pi \frac{d}{N} i) = \sin(\frac{2\pi}{N}(I_i + e_i)) ,$$

here $N = 2^m$, the integer part I_i is the actual LUT address, and the fractional part $e_i = \mathrm{mod}(A_i / 2^{r-m}, 1) \in (0,1)$ is the phase error sequence.

PT-error sequence e_i varies periodically and results from the sampling of a hipothetical sawtooth waveform with a period of $1/\mathrm{mod}(d,1)$, measured in clock cycles. Hence the fundamental *numerical frequency* of phase error e_i is

$$f_e / f_c = \mathrm{mod}(d,1) = \varepsilon / 2^r < 1 ,$$

where ε can be considered as a (hipothetical) "tuning word" for phase error (and this *integer* data will be used to compute the PT-spur locations). Since $\mathrm{mod}(d,1) = L/E$, therefore the *numerical period of PT-error sequence* is E (a factor of 2).

Using small angle approximation for realistic NCO implementations ($N = 2^m >> 1$ and disregarding scaling constant)

$$x_i \approx \sin(\frac{2\pi}{N} I_i) + \frac{2\pi}{N} e_i \cos(\frac{2\pi}{N} I_i) = T_i + ps_i \ .$$

Consequently, the high precision NCO output $T_i \approx x_i - ps_i$ will be composed of the desired clean x_i sinewave corrupted by ps_i PT-spurs, a cosine modulated harmonics of the $(2\pi/N) \cdot e_i$ sawtooth waveform. Spur components are symmetrically located *in pairs* around the desired signal component, and it follows from the E periodicity of phase error sequence that the <u>number</u> of PT-spurs is E–1.

At a given E, varying whether V or L (prime to a given E) will just *permute* the same spectral lines (only relative location of the signal and PT-spurs varies but not the levels).

Due to aliasing and taking into account the modulation theorem of Fourier transform, the <u>locations</u> of PT-spur (overlapped modulation structure)

$$z_h = \mathrm{mod}(D + h \cdot \varepsilon, S), \quad S = 2^r \text{ and } h = 0, 1, 2 \ldots E-1$$

$$sp_h = \mathrm{if}\ (z_h < (S/2), z_h, (S - z_h)).$$

Here $h = 0$ is the signal component. The analog frequency of PT-spurs is $fp_h = (sp_h / S) \cdot f_c$.

The <u>levels</u> of PT-spur are also calculated by an FFT transform taken at the NCO output.

Some of the special cases: (1) There are no PT-spurs if $E = 1$, and zeros in the truncated low r–m phase accumulator bits indicate these favourable cases. (These tuning states are suitable to *test the AQ-spurs only* in NCO. Full testing of the AQ-spurs is only possible with simulation. Similarly, testing separately the PT-spurs is only possible with *simulation* using a high precision T_i table output.) (2) The worst case occurs when $E = 2$ (independently of V), in these cases spur energy apparenly "concentrates" on one PT-spur. A simple estimate of bound on maximum PT-spur component to signal ratio is SpSR ≈ $(-6 \cdot m + 7)$ dBc. As a general rule, LUT m-bit address length is 2 bits higher than the n-bit data precision to make the highest PT-spur component smaller than the background AQ "noise floor". (3) There are many PT-spurs in the other limiting cases, if E is long.

IV. EXAMPLES OF SIMULATION

The two sources of numerical distortion (AQ and PT) appear simultaneously and they possibly exercise mutual influence on each other. Simulations help to discover the composed spectral maps for specific applications and tuning ranges (e.g. for the proper selection of the data before parameterization of an NCO core in FPGA).

A set of *interactive* Mathcad worksheets based on structural relations described above was developed that exemplified some topics of DDS numerical distortion. One can reach some of them online at the URL:

www.hit.bme.hu/people/papay/sci/DDS/simul.htm,

where a freely downloadable interactive viewer is also provided. Here only two illustrative examples are given, as a drop in the bucket, to demonstrate the method. The first one illustrates an interaction between AQ and PT (Fig. 2), while the second one presents a multi-sine (Fig. 3). All spectra were computed by 8K FFT with BH7 (a 7-term Blackman-Harris) window.

As the desired f/f_c is the ratio of small integers but the denominator is not a power of 2, the *actual* (tunable) numerical frequency has a "small offset" (because of finite frequency resolution), hence spurs will concentrate close to the fundamental frequency and near some other lines (Fig. 2a). Modifying only the data precision (n-bit: 10 → 7), an unforeseen interaction occurs (Fig. 2b).

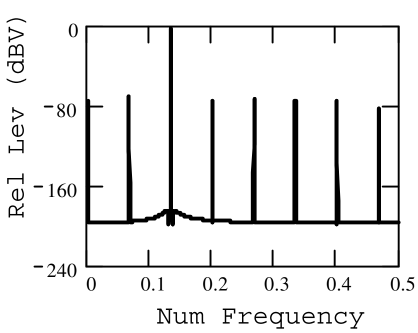

Fig. 2a. Spectra of a synthesized single-tone (sinusoid). Simulation parameters: r = 32, m = 12, **n = 10** and the desired $f/f_c = 2/15$ (i.e. actual D is all two in hexa).

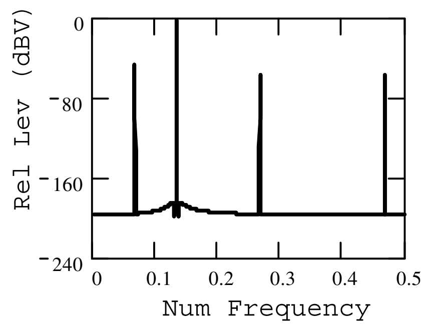

Fig. 2b. Same as Fig. 2a, but with limited data precision: **n = 7**.

In multi-tone case, the simple superposition is invalid because of the strong nonlinearities and only simulation proves to be the usable tool (Fig. 3, where the levels of harmonically related components change with –5 dB). The limited memory capacity (1K, i.e. m-bit: 10) causes unpleasant spurs, which can be lowered significantly by choosing a higher m-bit (i.e. by increasing the waveform length, as it would be the case in many DDS-based ARB generators).

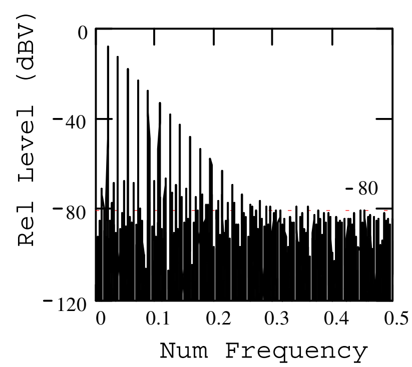

Fig. 3. Spectra of synthesized multi-tone with limited (1K) memory. Simulation parameters: r = 48, **m = 10**, n = 12 and the desired fundamental f/f_c = 0.7/40.

To summarize, although our description was mainly restricted to a single-tone case, by simulations the investigations can further be extended to other practical cases, as well.

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