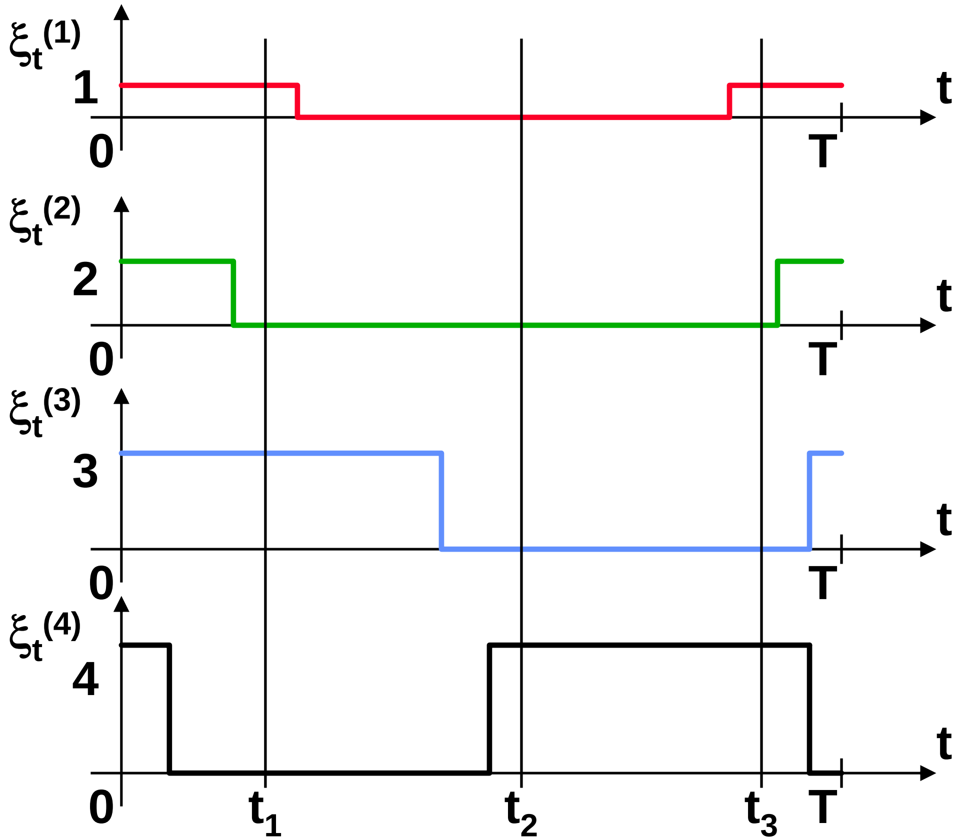
Pr.11. Stochastic process having 4 realisations in the range of (0,T) with given probabilities:



$p_1=0.1$

$p_2=0.2$

$p_3=0.3$

$p_4=0.4$

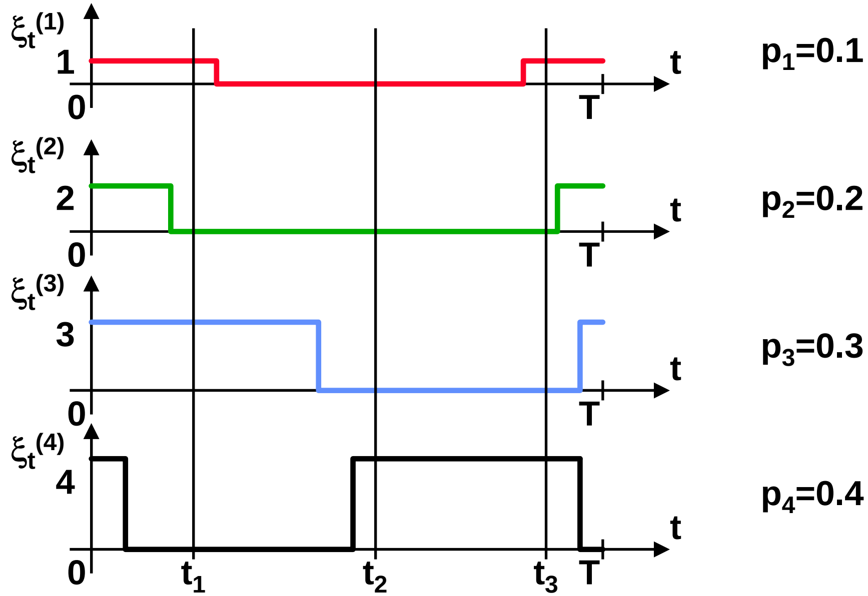
a) Stationarity?

b) Correlation function's values?

c) Probability distribution function's values?

d) Joint (2 dimensions) PDF values?

Pr. 1. a) Stationarity



a1) Wide-sense stationarity?

$$\mathbf{M}\,\xi_t = \mathbf{M}\,\xi_0,$$

$$\mathbf{M}\,\xi_t\,\xi_\vartheta = \mathbf{M}\,\xi_{\vartheta - t}\,\xi_0 = \mathbf{M}\,\xi_\tau\,\xi_0,$$

$$\mathbf{M}(\xi_{t_i}) = \sum_{k=1}^{n} p_k \cdot x_{t_i}^{(k)}$$

$$\mathbf{M}(\xi_{t_1}) = \sum_{k=1}^{n} p_k \cdot x_{t_1}^{(k)} =$$

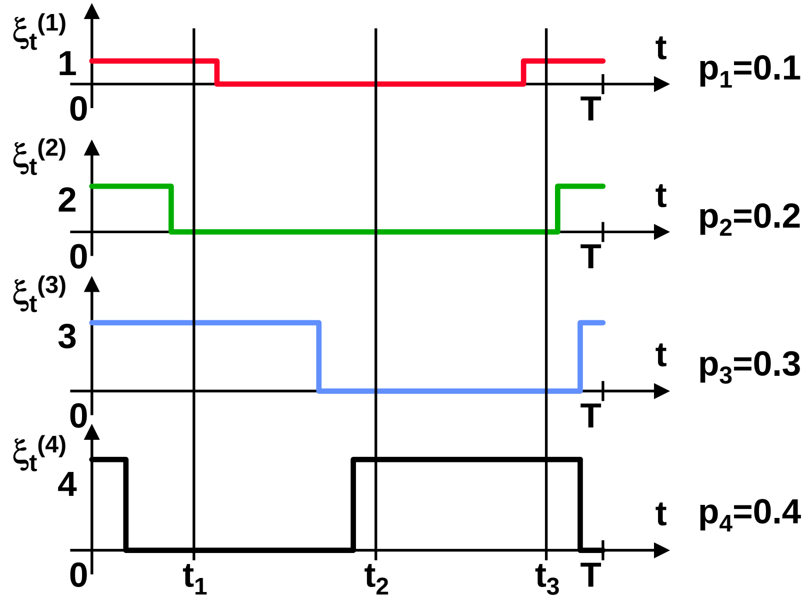
$$= p_1 \cdot x_{t_1}^{(1)} + p_2 \cdot x_{t_1}^{(2)} + p_3 \cdot x_{t_1}^{(3)} + p_4 \cdot x_{t_1}^{(4)} =$$

$$= 0.1 \cdot 1 + 0.3 \cdot 3 = 1$$

$$\mathbf{M}(\xi_{t_2}) = \sum_{k=1}^{n} p_k \cdot x_{t_2}^{(k)} = 0.4 \cdot 4 = 1.6$$

The process is nonstationary in any sense!

Pr. 1. b) Correlation function



b1) Let t_1 and t_2

$$\boxed{L_\xi(t,\vartheta)=\mathbf{M}\xi_t\xi_\vartheta}$$

$$L_\xi(t,\vartheta)=\mathbf{M}\xi_t\xi_\vartheta=\sum_{k=1}^{n}p_k\cdot x_t^{(k)}\cdot x_\vartheta^{(k)}=$$

$$L_\xi(t_1,t_2)=\mathbf{M}\xi_{t_1}\xi_{t_2}=\sum_{k=1}^{n}p_k\cdot x_{t_1}^{(k)}\cdot x_{t_2}^{(k)}=$$

$$=p_1\cdot x_{t_1}^{(1)}\cdot x_{t_2}^{(1)}+p_2\cdot x_{t_1}^{(2)}\cdot x_{t_2}^{(2)}+p_3\cdot x_{t_1}^{(3)}\cdot x_{t_2}^{(3)}+p_4\cdot x_{t_1}^{(4)}\cdot x_{t_2}^{(4)}=$$

$$=0.1\cdot 1\cdot 0+0.2\cdot 0\cdot 0+0.3\cdot 3\cdot 0+0.4\cdot 0\cdot 4=0$$

b2) Let t_1 and t_3

$$L_\xi(t_1,t_3)=\mathbf{M}\xi_{t_1}\xi_{t_3}=\sum_{k=1}^{n}p_k\cdot x_{t_1}^{(k)}\cdot x_{t_3}^{(k)}=$$

$$=p_1\cdot x_{t_1}^{(1)}\cdot x_{t_3}^{(1)}+p_2\cdot x_{t_1}^{(2)}\cdot x_{t_3}^{(2)}+p_3\cdot x_{t_1}^{(3)}\cdot x_{t_3}^{(3)}+p_4\cdot x_{t_1}^{(4)}\cdot x_{t_3}^{(4)}=$$

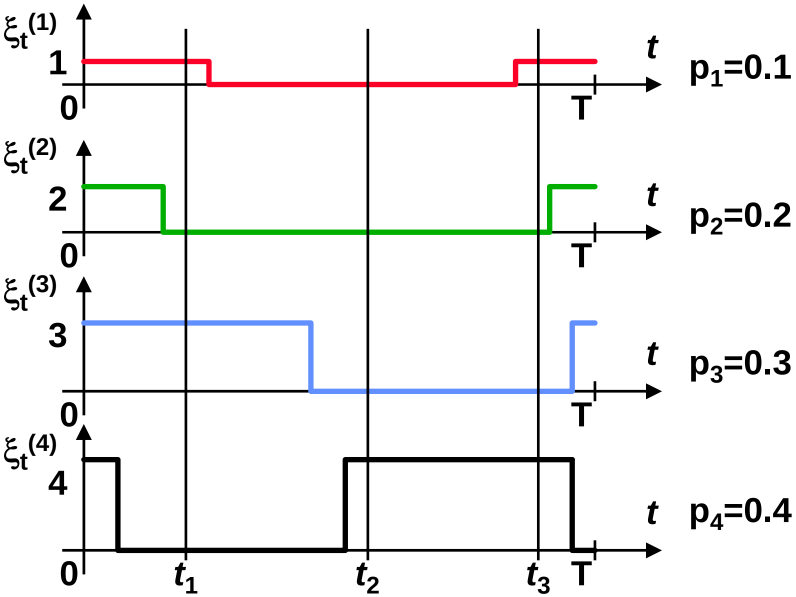
$$=0.1\cdot 1\cdot 1+0.2\cdot 0\cdot 0+0.3\cdot 3\cdot 0+0.4\cdot 0\cdot 4=0.1$$

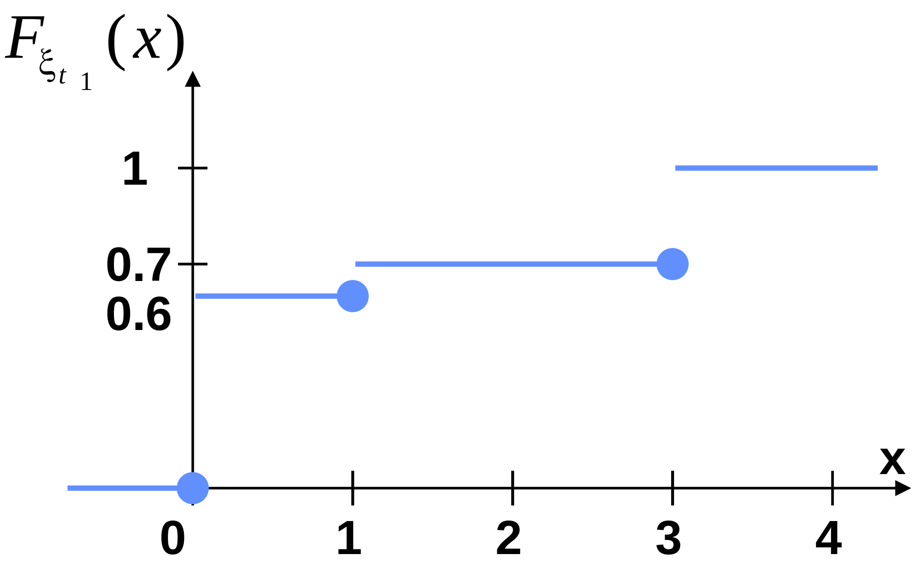
Pr. 1. c) Probability distribution function

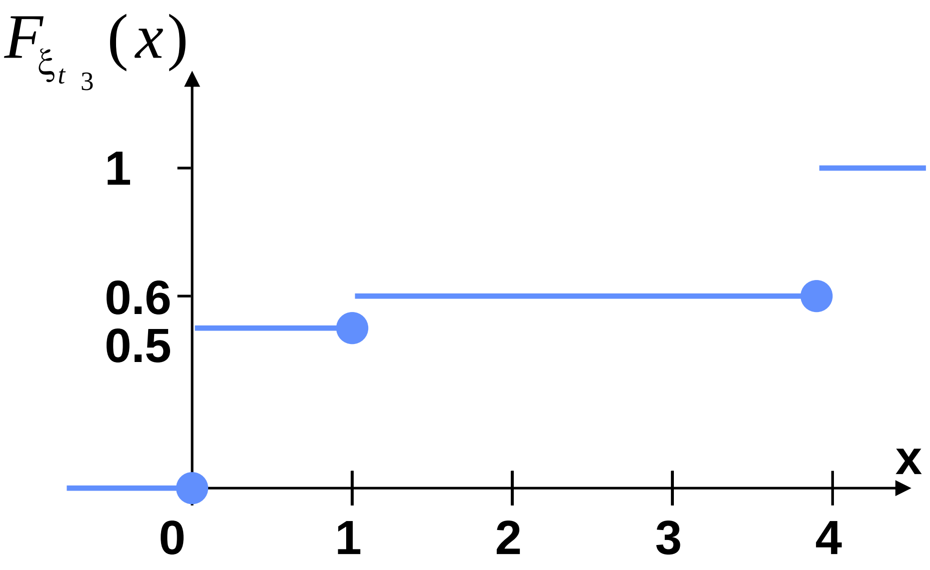
c1) Let t_1

$$F_{\xi_{t_1}}(x) = \mathrm{P}(\xi_{t_1} < x) =$$

$$= \begin{cases} \text{if } x \le 0, \text{ then } 0 \\ \text{if } 0 < x \le 1, \text{ then } p_2 + p_4 = 0.6 \\ \text{if } 1 < x \le 3, \text{ then } p_1 + p_2 + p_4 = 0.7 \\ \text{if } x > 3, \text{ then } p_1 + p_2 + p_3 + p_4 = 1 \end{cases}$$





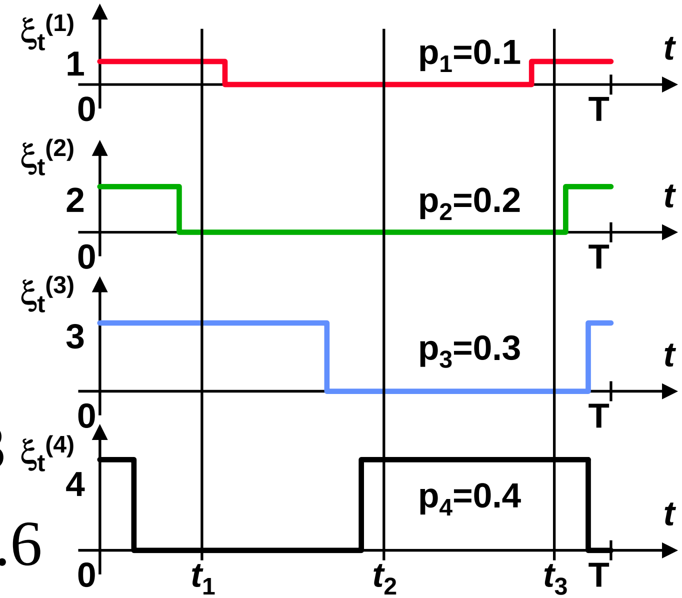


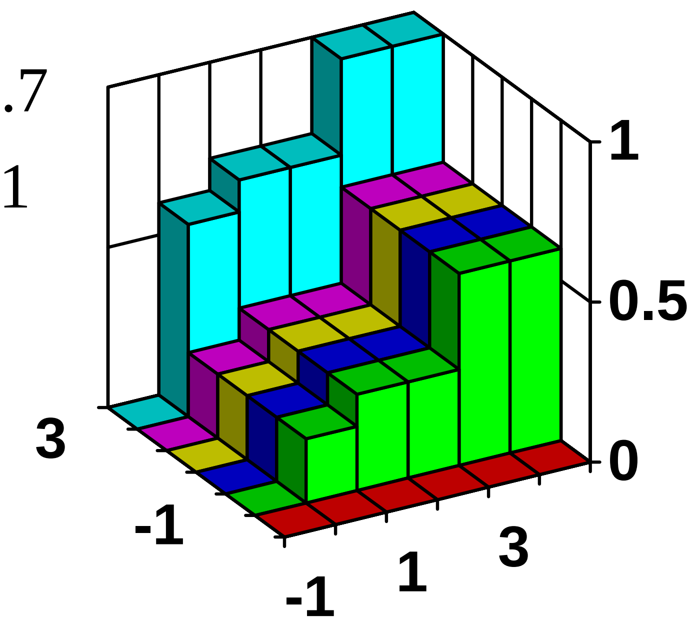
Pr. 1. d) Joint PDF (for 2 dimensions)

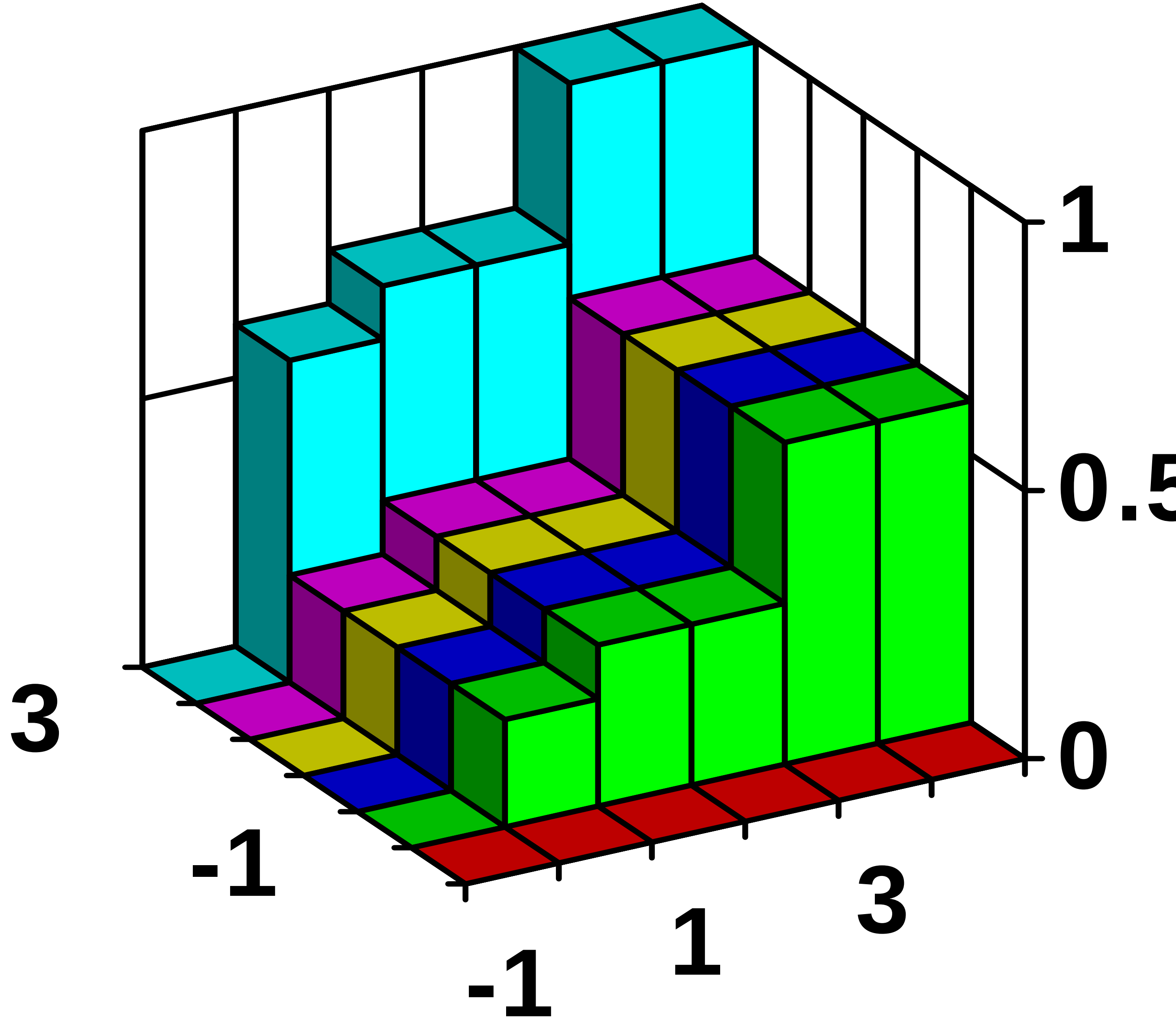
d1) Let t_1 and t_2

$$F_{\xi_{t_1},\xi_{t_1}}(x_1,x_2)=\mathrm{P}(\xi_{t_1}<x_1,\xi_{t_2}<x_2)=$$

$$=\begin{cases}\text{if } x_1\le 0\bigcap x_2\le 0,\text{ then } 0\\ \text{if } 0<x_1\le 1\bigcap 0<x_2\le 4,\text{ then } p_2=0.2\\ \text{if } 1<x_1\le 3\bigcap 0<x_2\le 4,\text{ then } p_1+p_2=0.3\\ \text{if } x_1>3\bigcap 0<x_2\le 4,\text{ then } p_1+p_2+p_3=0.6\\ \text{if } 0<x_1\le 1\bigcap x_2>4,\text{ then } p_2+p_4=0.6\\ \text{if } 1<x_1\le 3\bigcap x_2>4,\text{ then } p_1+p_2+p_4=0.7\\ \text{if } x_1>3\bigcap x_2>4,\text{ then } p_1+p_2+p_3+p_4=1\end{cases}$$






1
0.5
0
3
-1
-1
1
3

Pr.2. Investigate the stationarity of stochastic processes:

a) $\xi_t = \mathrm{U}_0 \cdot \sin \Omega t, \text{ where } \Omega$

is a uniformly distributed random variable over (0.1 B , B) and U_0 is a constant

b) $\xi_t = \eta \cdot \cos \mathrm{F} t, \text{ where } \eta$

is a uniformly distributed random variable over (0 , 2) and F is a constant

Pr.13. Let $\{\xi_t, t \in (-\infty, \infty)\}$ a wide sense stationary process, and Φ an independent random variable, uniformly distributed over (0, 2π]. Calculate the correlation function of process η_t , if

$$\eta_t = \xi_t \cdot \cos(\Omega t + \Phi), \text{and } R_\xi(\tau) \text{ is given.}$$

Solution:

$$L_\eta(t, \vartheta) = \mathbf{M}[\xi_t \cdot \cos(\Omega t + \Phi) \cdot \xi_\vartheta \cdot \cos(\Omega \vartheta + \Phi)] =$$

$$= \mathbf{M}[\xi_t \cdot \xi_\vartheta \cdot \cos(\Omega t + \Phi) \cdot \cos(\Omega \vartheta + \Phi)] \overset{indpt}{=}$$

$$= \mathbf{M}[\xi_t \cdot \xi_\vartheta] \cdot \mathbf{M}[\cos(\Omega t + \Phi) \cdot \cos(\Omega \vartheta + \Phi)] =$$

$$= R_\xi(t - \vartheta) \cdot \mathbf{M}[\cos(\Omega t + \Phi) \cdot \cos(\Omega \vartheta + \Phi)] =$$

$$= R_\xi(\tau) \cdot \mathbf{M}[\cos(\Omega t + \Phi) \cdot \cos(\Omega \vartheta + \Phi)]$$

Let us calculate the mean:

$$L_v(t,\vartheta) = \frac{\mathrm{U}^2}{2}\cdot\int_0^{2\pi}\{\cos[\Omega(t+\vartheta)+2\Phi]+\cos[\Omega(t-\vartheta)]\}\cdot\frac{1}{2\pi}\cdot d\Phi =$$

$$= \frac{\mathrm{U}^2}{4\pi}\cdot[0+2\pi\cdot\cos\Omega(t-\vartheta)] = \frac{\mathrm{U}^2}{2}\cdot\cos\Omega(t-\vartheta)$$

Conclusion:

$$L_v(t,\vartheta) = R_v(\tau) = \frac{\mathrm{U}^2}{2}\cdot\cos\Omega\tau$$

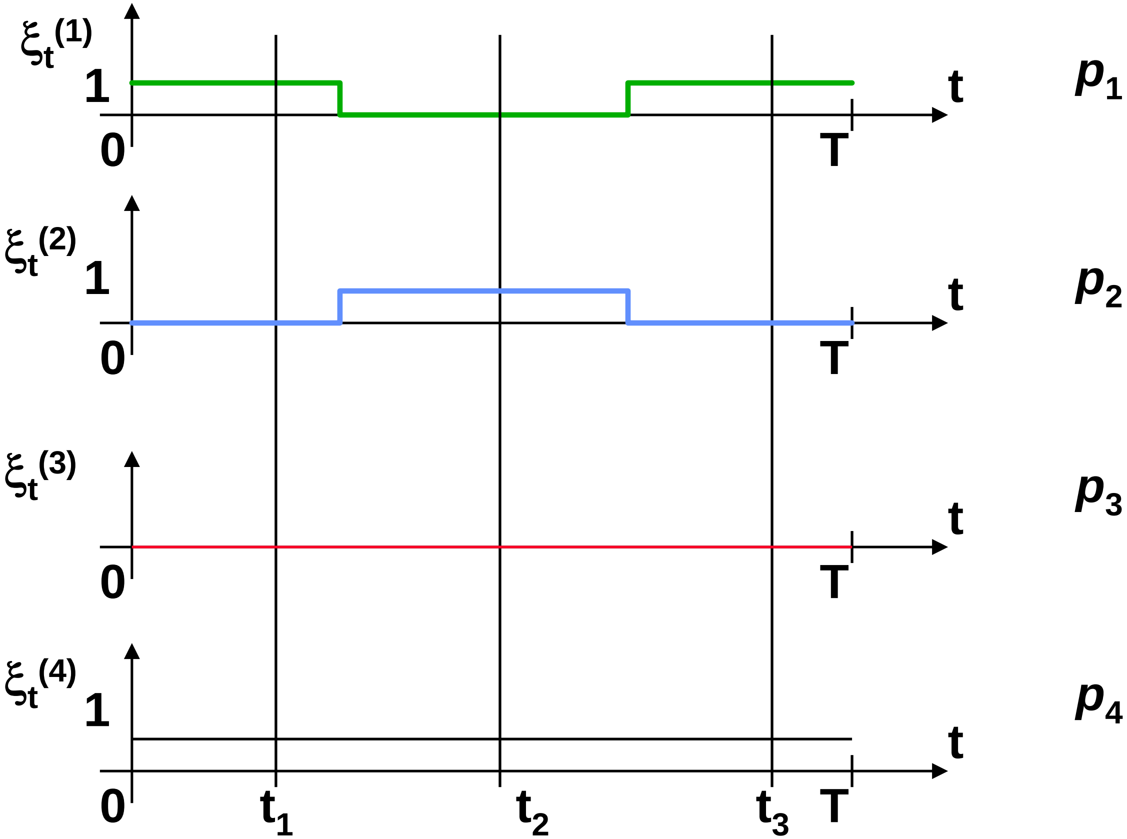
Going back to η:

$$L_\eta(t,\vartheta) = R_\xi(\tau)\cdot\mathbf{M}[\cos(\Omega t+\Phi)\cdot\cos(\Omega\vartheta+\Phi)] =$$

$$= R_\eta(\tau) = \frac{1}{2}\cdot R_\xi(\tau)\cdot\cos\Omega\tau$$

Check the stationarity of η!

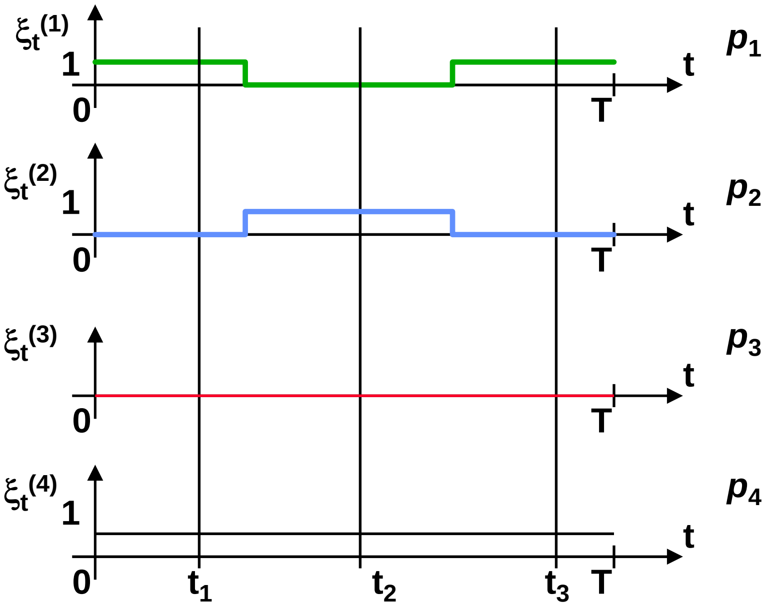
Pr.4. Let ξ_t a stochastic process given by its sample functions:



a) What probabilities of realizations can result in the independence of random variables ξ_1 and ξ_2 ?

a) What conditions are needed for independence?

joint probabilities must be equal to the multiplication of standalone ones



The two random variables ξ_1 and ξ_2 are binary, thus their joint events have four values, namely: 00, 01, 10 and 11

The condition of independence:

$$P(\xi_{t1} = 0 \cap \xi_{t2} = 0) = P(\xi_{t1} = 0) \cdot P(\xi_{t2} = 0)$$

$$P(\xi_{t1} = 0 \cap \xi_{t2} = 1) = P(\xi_{t1} = 0) \cdot P(\xi_{t2} = 1)$$

$$P(\xi_{t1} = 1 \cap \xi_{t2} = 0) = P(\xi_{t1} = 1) \cdot P(\xi_{t2} = 0)$$

$$P(\xi_{t1} = 1 \cap \xi_{t2} = 1) = P(\xi_{t1} = 1) \cdot P(\xi_{t2} = 1)$$

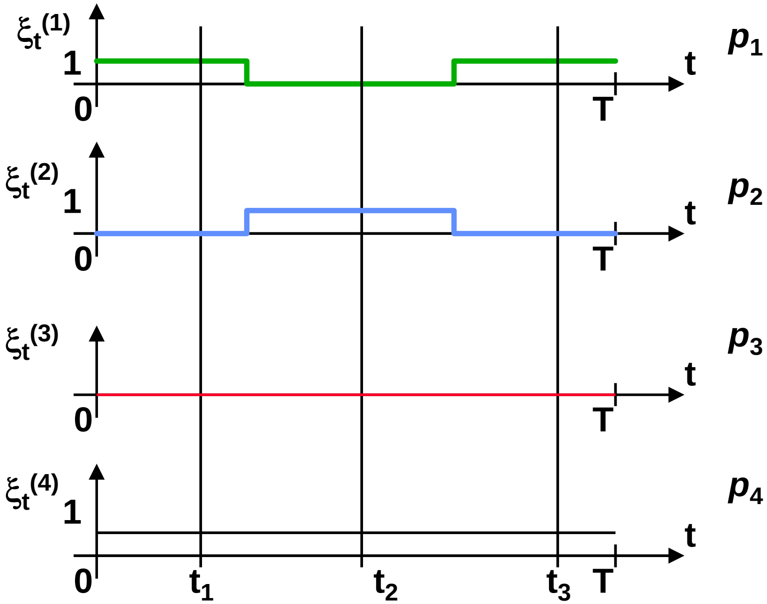
Calculate the probabilities occuring in these equations

$$P(\xi_{t1}=0 \cap \xi_{t2}=0)=P(\xi_{t1}=0)\cdot P(\xi_{t2}=0)$$

$$P(\xi_{t1}=0 \cap \xi_{t2}=1)=P(\xi_{t1}=0)\cdot P(\xi_{t2}=1)$$

$$P(\xi_{t1}=1 \cap \xi_{t2}=0)=P(\xi_{t1}=1)\cdot P(\xi_{t2}=0)$$

$$P(\xi_{t1}=1 \cap \xi_{t2}=1)=P(\xi_{t1}=1)\cdot P(\xi_{t2}=1)$$



Joint probabilities:

$$P(\xi_{t1}=0 \cap \xi_{t2}=0)=p_3$$

$$P(\xi_{t1}=0 \cap \xi_{t2}=1)=p_2$$

$$P(\xi_{t1}=1 \cap \xi_{t2}=0)=p_1$$

$$P(\xi_{t1}=1 \cap \xi_{t2}=1)=p_4$$

Standalone probabilities:

$$P(\xi_{t1}=0)=p_2+p_3$$

$$P(\xi_{t1}=1)=p_1+p_4$$

$$P(\xi_{t2}=0)=p_1+p_3$$

$$P(\xi_{t2}=1)=p_2+p_4$$

Substitutung into the equations:

$$P(\xi_{t1} = 0 \bigcap \xi_{t2} = 0) = P(\xi_{t1} = 0) \cdot P(\xi_{t2} = 0)$$

$$P(\xi_{t1} = 0 \bigcap \xi_{t2} = 1) = P(\xi_{t1} = 0) \cdot P(\xi_{t2} = 1)$$

$$P(\xi_{t1} = 1 \bigcap \xi_{t2} = 0) = P(\xi_{t1} = 1) \cdot P(\xi_{t2} = 0)$$

$$P(\xi_{t1} = 1 \bigcap \xi_{t2} = 1) = P(\xi_{t1} = 1) \cdot P(\xi_{t2} = 1)$$

we get:

$$p_3 = (p_2 + p_3) \cdot (p_1 + p_3)$$

$$p_2 = (p_2 + p_3) \cdot (p_2 + p_4)$$

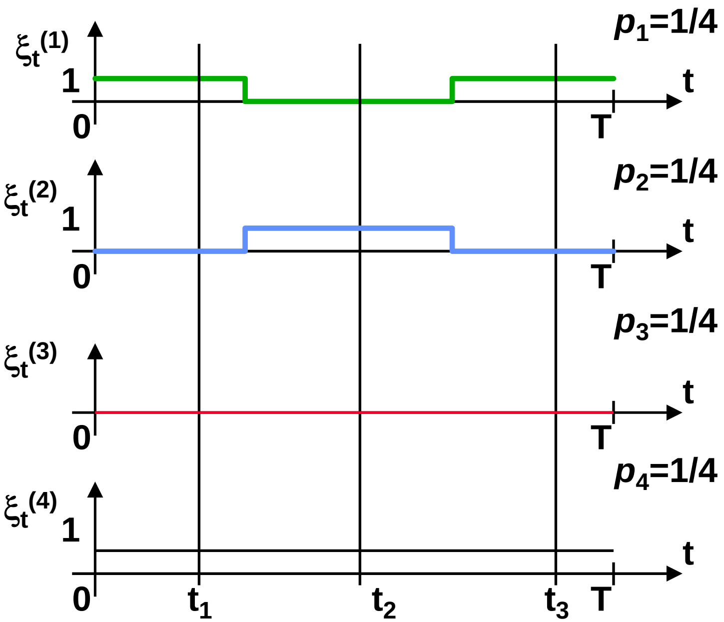
$$p_1 = (p_1 + p_4) \cdot (p_1 + p_3)$$

$$p_4 = (p_1 + p_4) \cdot (p_2 + p_4)$$

and a possible solution?

b) Check the stationarity of ξ_t !

(using $p_1=p_2=p_3=p_4=1/4$)



First we check mean values at different times:

$$\mathbf{M}\,\xi_{t_1} = p_1 \cdot 1 + p_4 \cdot 1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\mathbf{M}\,\xi_{t_2} = p_2 \cdot 1 + p_4 \cdot 1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

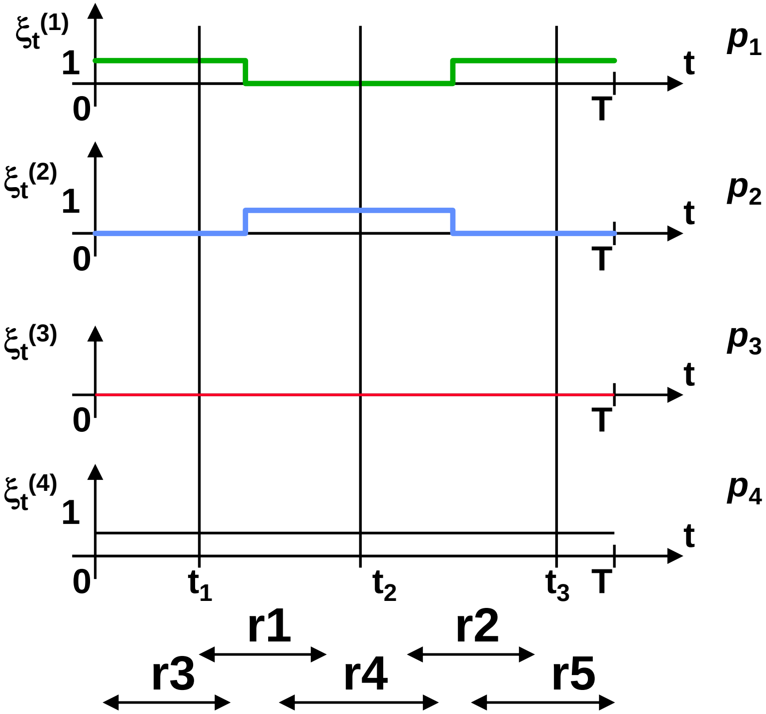
$$\mathbf{M}\,\xi_{t_3} = p_1 \cdot 1 + p_4 \cdot 1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Is this process eventually stationary?

To get an answer,

we should check the correlation function

taking the correlation, for example, over a time difference of less then T/3, five different ranges can be found:



for r1

$$\mathbf{M}\,\xi_{t_1}\cdot\xi_{t_2} = p_4\cdot 1\cdot 1 = 1/4$$

for r2

$$\mathbf{M}\,\xi_{t_2}\cdot\xi_{t_3} = p_4\cdot 1\cdot 1 = 1/4$$

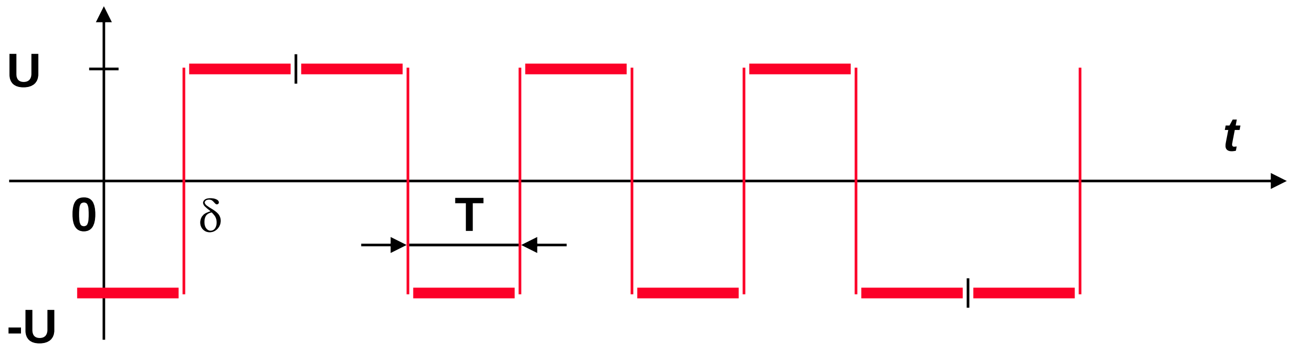
but for r3, r4 and r5

$$\mathbf{M}\,\xi_{t_i}\cdot\xi_{t_j} = 1/2$$

Thus we can conlude: the process is nonstationary.

Pr. 21. Calculate the spectral density of "random data" signal, or random binary wave

- **It is a stochastic process characterized by a typical sample function:**



- **The values U and -U correspond to 0 and 1 as a series of independent random binary variables determined by tossing a fair coin**
- **The beginning of a starting point of T is delayed by δ that is a random variable, uniformly distributed over [0, T)**

- **How can we calculate spectral density?**
  - **first: we determine correlation function**
  - **second: take the Fourier transform of it**

$$R_\xi(t,\vartheta) = \mathbf{M}\xi_t\,\xi_\vartheta = \sum_{x_1=-\mathrm{U}}^{\mathrm{U}}\sum_{x_2=-\mathrm{U}}^{\mathrm{U}} x_1 \cdot x_2 \cdot P\big[(\xi_t = x_1)\cap(\xi_\vartheta = x_2)\big] =$$

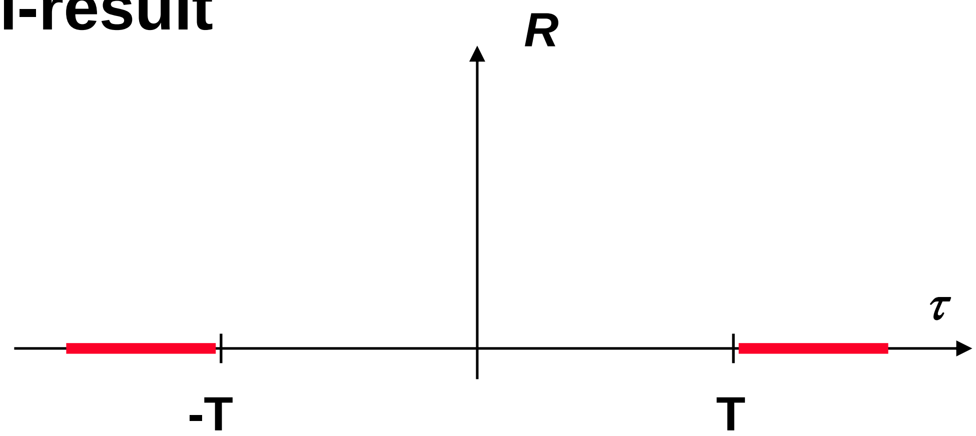
$$= \mathrm{U}^2 \cdot \Big\{P\big[(\xi_t = \mathrm{U})\cap(\xi_\vartheta = \mathrm{U})\big] + P\big[(\xi_t = -\mathrm{U})\cap(\xi_\vartheta = -\mathrm{U})\big]\Big\} -$$

$$- \mathrm{U}^2 \cdot \Big\{P\big[(\xi_t = -\mathrm{U})\cap(\xi_\vartheta = \mathrm{U})\big] + P\big[(\xi_t = \mathrm{U})\cap(\xi_\vartheta = -\mathrm{U})\big]\Big\}$$

- **Calculate the function for** $|t-\vartheta| > \mathrm{T}$
- **In this case the random variables ξ_t and ξ_θ are independent and the mean value is 0**

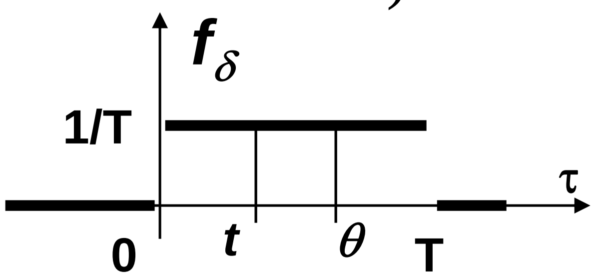
$$R_\xi(t,\vartheta) = \mathbf{M}\,\xi_t \cdot \xi_\vartheta \underset{|t-\vartheta|>\mathrm{T}}{\overset{indpdt}{=}} \mathbf{M}\,\xi_t \cdot \mathbf{M}\,\xi_\vartheta = 0$$

- **The semi-result**



- **What about between -T and T ?**
- **If** $t - \vartheta \leq \mathrm{T}$ **then** $P(\text{edge of T is between } t \text{ and } \vartheta) = ?$
- **Since δ is**

uniformly distributed over [0, T)

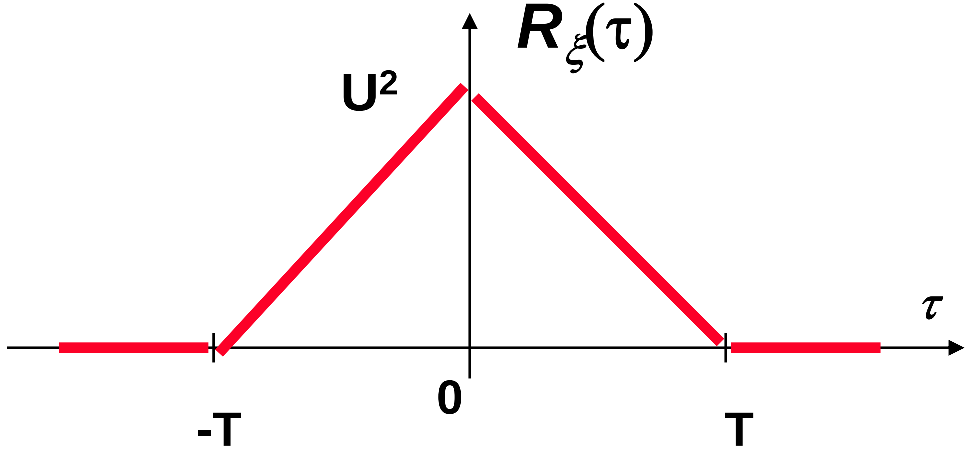


$$P(\text{edge of T is between } t \text{ and } \vartheta) = P(t < \delta < \vartheta) = \frac{|t - \vartheta|}{\mathrm{T}}$$

- **If there is no edge between *t* and θ then the samples are the same**
- **thus**

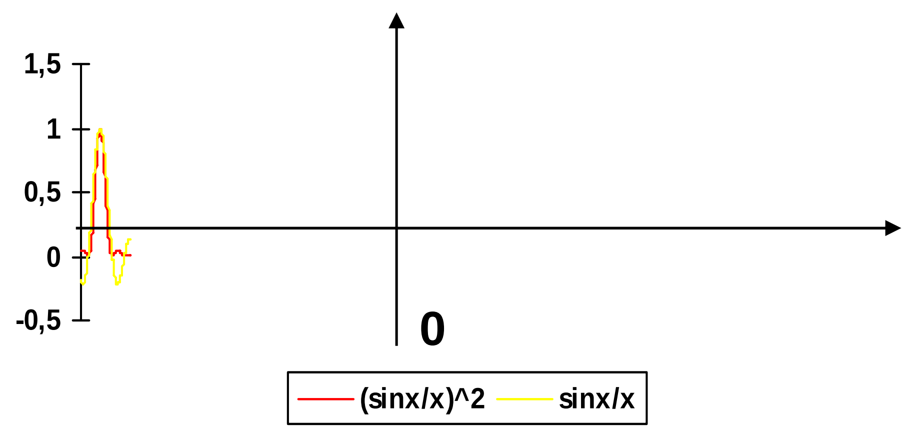
$$R_\xi(t,\vartheta) = \mathbf{M}\xi_t \cdot \xi_\vartheta \underset{|t-\vartheta|\le \mathrm{T}}{=} \mathrm{U}^2 \cdot \left(1 - \frac{|t-\vartheta|}{\mathrm{T}}\right)$$

- **The final result**

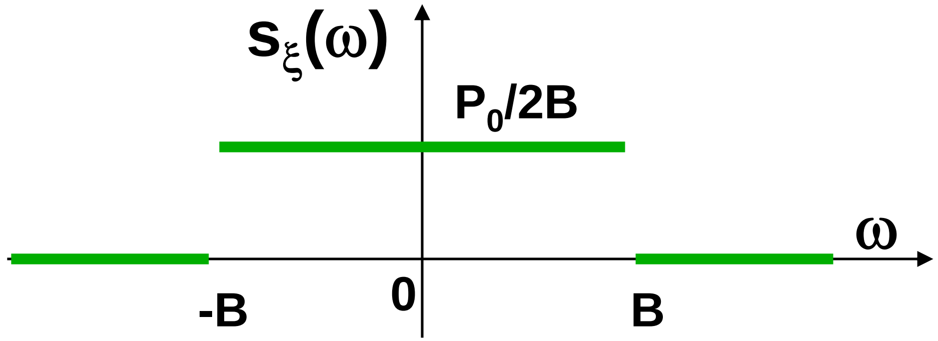


- **The spectral density is the Fourier transform ... =**

$$s_\xi(\omega) = \frac{\mathrm{U}^2 \cdot \mathrm{T}}{2\pi} \left(\frac{\sin \frac{\omega \mathrm{T}}{2}}{\frac{\omega \mathrm{T}}{2}} \right)^2$$



Pr. 22. Calculate the correlation function of random process ξ_t if $s_\xi(\omega)$ is given:



- the process is named as *band-limited* with constant spectral density
- correlation function

$$R_\xi(\tau) = \int_{-\infty}^{\infty} s_\xi(\omega) \cdot \mathrm{e}^{\mathrm{j}\omega\tau} d\omega = \frac{\mathrm{P}_0}{2\mathrm{B}} \int_{-\mathrm{B}}^{\mathrm{B}} \mathrm{e}^{\mathrm{j}\omega\tau} d\omega =$$

$$= \frac{\mathrm{P}_0}{2\mathrm{B}} \cdot \frac{\mathrm{e}^{\mathrm{jB}\tau} - \mathrm{e}^{-\mathrm{jB}\tau}}{\mathrm{j}\tau} = \mathrm{P}_0 \frac{\sin \mathrm{B}\tau}{\mathrm{B}\tau}$$