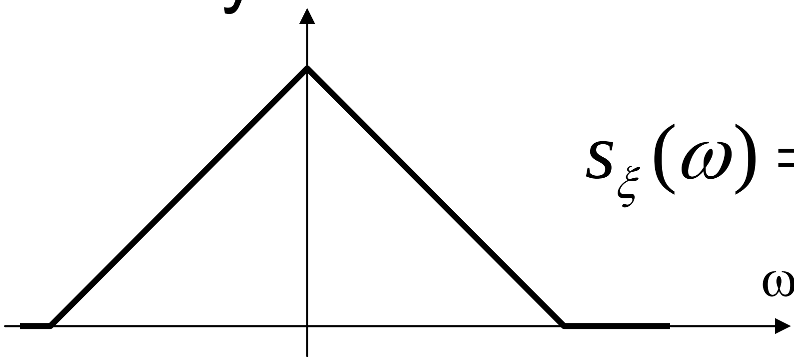
Compression coding

- Let us have a stochastic process with spectral density function:



$$s_\xi(\omega) = \begin{cases} s_0 - \left|\frac{\omega}{B}\right| \cdot s_0, & if \ \omega \in [-B, B] \\ 0, & elsewhere \end{cases}$$

$$\bar{s}_\xi = \frac{1}{2B} \cdot B \cdot s_0 = \frac{s_0}{2}; \quad \frac{s_\xi(\omega)}{\bar{s}_\xi} = \frac{s_0\left(1 - \frac{|\omega|}{B}\right)}{s_0/2} = 2 \cdot \left(1 - \frac{|\omega|}{B}\right)$$

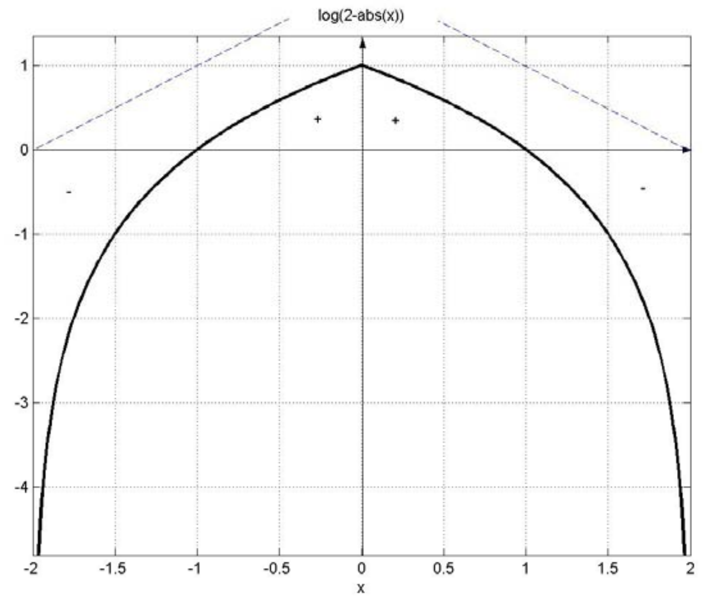
- Calculate the prediction gain and the number of spare-able binary digits by KL transformation!

$$\frac{1}{2}\cdot\log_2(G_p)=$$

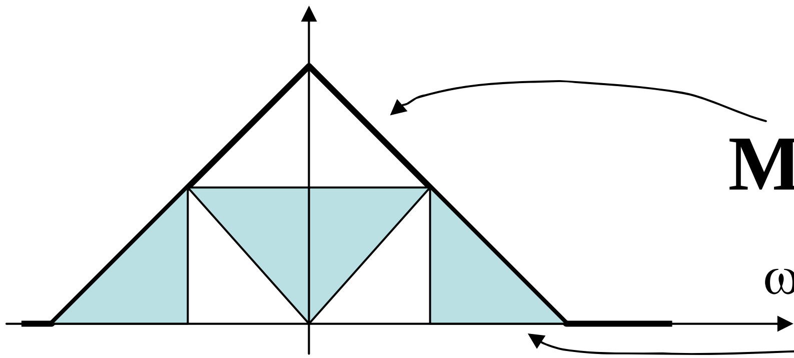
$$=-\frac{1}{2}\cdot\frac{1}{2B}\int_{-B}^{B}\log_2\frac{s_\xi(\omega)}{\overline{s}_\xi}d\omega=$$

$$=-\frac{1}{2}\cdot\frac{1}{2B}\int_{-B}^{B}\log_2\left(2\left(1-\frac{|\omega|}{B}\right)\right)d\omega=$$

$$=-\frac{1}{2}+\frac{1}{2\cdot\ln(2)}=0.2213$$



- Calculate the # of binary digits spared by sub-band coding, when using 2 sub-bands only!

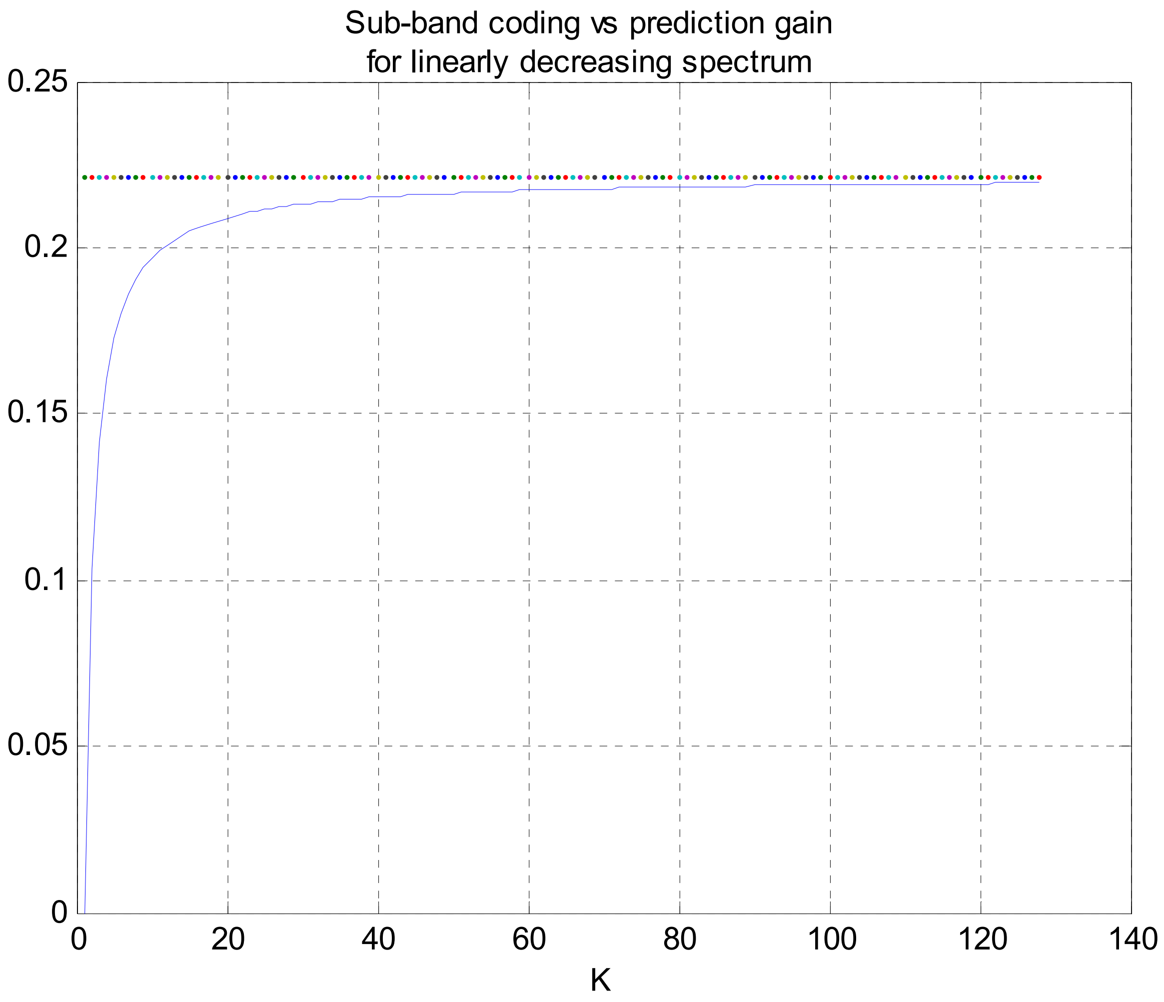


$$\mathbf{M}\xi_1^2 = 3\cdot\frac{Bs_0}{4};\mathbf{M}\xi_2^2 = \frac{Bs_0}{4}$$

$$\frac{1}{2}\cdot\log_2\frac{\frac{1}{K}\sum_{i=1}^{K}\mathbf{M}\xi_i^2}{\left(\prod_{i=1}^{K}\mathbf{M}\xi_i^2\right)^{\frac{1}{K}}} = \frac{1}{2}\cdot\log_2\frac{\frac{1}{2}\left(3\frac{Bs_0}{4}+\frac{Bs_0}{4}\right)}{\sqrt{3\frac{Bs_0}{4}\cdot\frac{Bs_0}{4}}} = \frac{1}{2}-\frac{1}{2}\cdot\log_2\sqrt{3} = 0.1038$$

$$K = 4$$

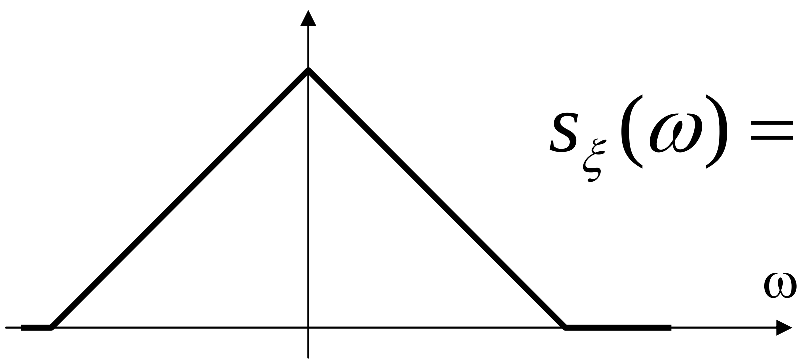
$$\frac{1}{2}\cdot\log_2\frac{\frac{1}{4}\left(\frac{7}{16}+\frac{5}{16}+\frac{3}{16}+\frac{1}{16}\right)}{\sqrt[4]{7\cdot 5\cdot 3\cdot 1\cdot\frac{1}{16}}} = \frac{1}{2}\cdot\log_2\frac{4}{\sqrt[4]{105}} = 1-\frac{1}{2}\cdot\log_2\sqrt[4]{105} = 0.161$$



Sub-band coding vs prediction gain
for linearly decreasing spectrum
0.25
0.2
0.15
0.1
0.05
0
0
20
40
60
80
100
120
140
K

Karhunen Loeve transformation

- Let us have a stochastic process with spectral density function:



$$s_\xi(\omega) = \begin{cases} s_0 - \left|\frac{\omega}{B}\right| \cdot s_0, & if \;\; \omega \in [-B, B] \\ 0, & elsewhere \end{cases}$$

- The correlation function of this process:

$$R_\xi(\tau) = Bs_0 \cdot \left(\frac{\sin(\frac{B}{2}\tau)}{(\frac{B}{2}\tau)} \right)^2 = Bs_0 \cdot \mathrm{sinc}^2(\tfrac{B}{2}\tau)$$

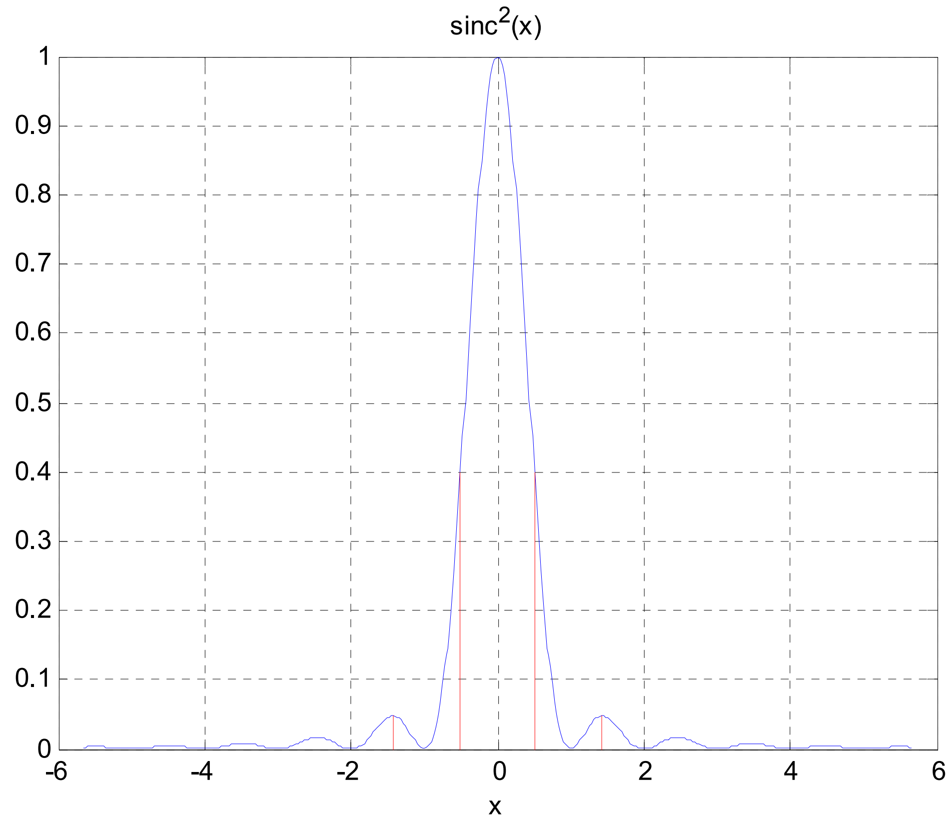
$$R_\xi(\tau) = Bs_0 \cdot \left(\frac{\sin(\frac{B}{2}\tau)}{(\frac{B}{2}\tau)} \right)^2 = Bs_0 \cdot \mathrm{sinc}^2(\tfrac{B}{2}\tau)$$

Samples should be taken with the interval of :

$$\tau_0 = \frac{\pi}{B} \Rightarrow$$

$$\Rightarrow \text{sampling points}$$

$$k \cdot \frac{B}{2} \cdot \tau_0 = k \cdot \frac{\pi}{2}$$



$$R_\xi(k \cdot \tau_0) = Bs_0 \cdot \mathrm{sinc}^2(k \tfrac{B}{2} \tfrac{\pi}{B}) = Bs_0 \cdot \mathrm{sinc}^2(k \tfrac{\pi}{2})$$

- The # of necessary binary symbols

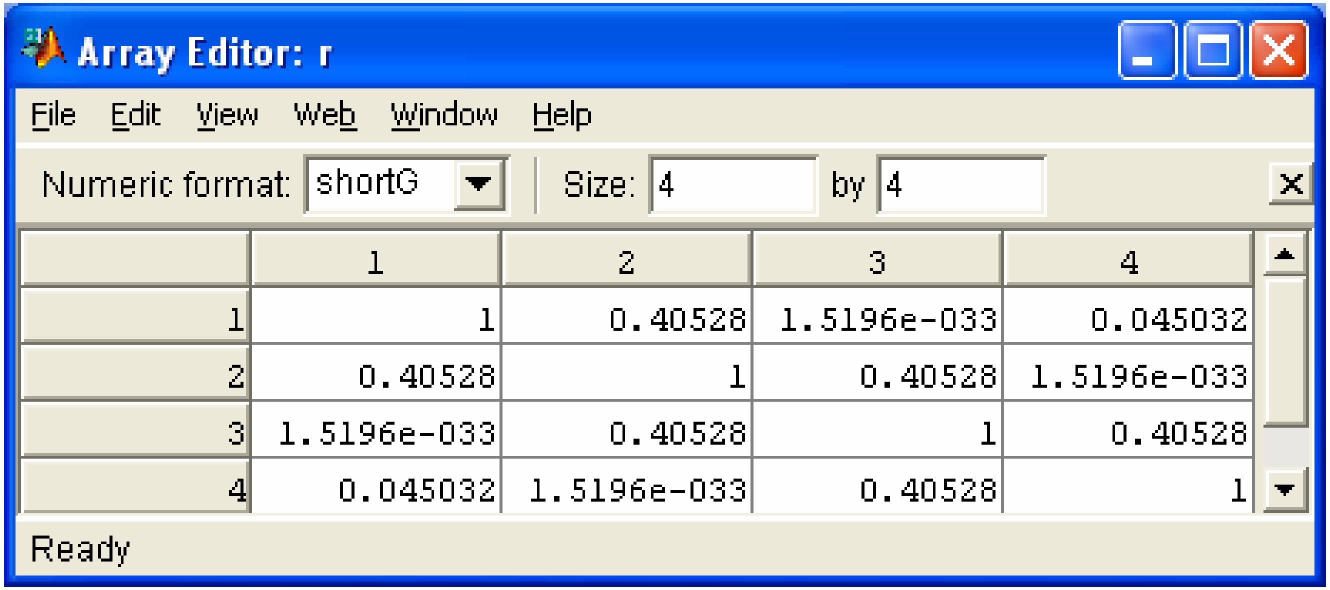
$$n = \frac{1}{2} \cdot ld(\rho) - \frac{1}{2} \cdot ld \frac{\frac{1}{K}\sum_{i=1}^{K} \lambda_i}{\left(\prod_{i=1}^{K} \lambda_i\right)^{1/K}}$$

The second term shows how many digits can be spared.
Calculating λ_i eigenvalues of R_ξ, we should construct it:

$$\mathbf{R}_\xi = \begin{pmatrix} R_\xi(0) & R_\xi(\pi/2) & R_\xi(\pi) & \cdots \\ R_\xi(-\pi/2) & R_\xi(0) & R_\xi(\pi/2) & \cdots \\ \cdots & & \ddots & \vdots \\ \cdots & R_\xi(-\pi) & R_\xi(-\pi/2) & R_\xi(0) \end{pmatrix}$$

- The rows of this matrix consist of shifted samples of correlation function

- For example *K*=4 (# of block elements)



	1	2	3	4
1	1	0.40528	1.5196e-033	0.045032
2	0.40528	1	0.40528	1.5196e-033
3	1.5196e-033	0.40528	1	0.40528
4	0.045032	1.5196e-033	0.40528	1

- Transformed blocks' correlation matrix

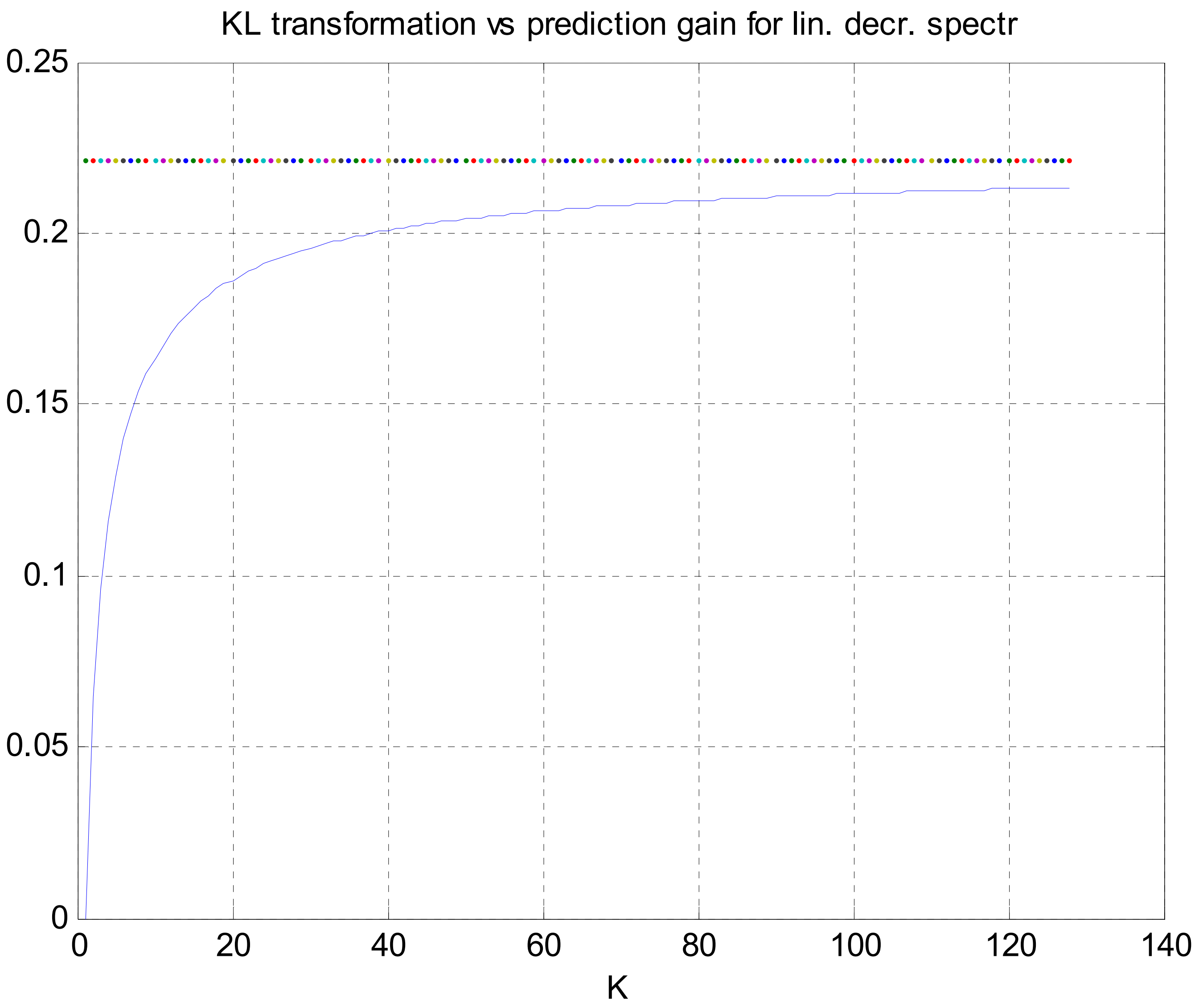
$$\mathbf{R}_\eta = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} = \begin{pmatrix} .331 & 0 & 0 & 0 \\ 0 & .782 & 0 & 0 \\ 0 & 0 & 1.22 & 0 \\ 0 & 0 & 0 & 1.67 \end{pmatrix}$$

- Calculating when the # of elements in blocks is 4

$$\frac{1}{2}\cdot ld\,\frac{\frac{1}{K}\sum_{i=1}^{K}\lambda_i}{\left(\prod_{i=1}^{K}\lambda_i\right)^{1/K}} = \frac{1}{2}\cdot ld\,\frac{\frac{1}{4}(.331+.782+1.22+1.67)}{(.331*.782*1.22*1.67)^{1/4}} =$$

$$= \frac{1}{2}\cdot ld\,\frac{\frac{1}{4}\cdot 4}{(.527)^{1/4}} = \frac{1}{2}\cdot ld\,\frac{1}{.852} = 0.116$$

- Next slide shows the result for longer blocks


KL transformation vs prediction gain for lin. decr. spectr
0.25
0.2
0.15
0.1
0.05
0
0
20
40
60
80
100
120
140
K

- Prediction coding of decreasing spectrum:
- Optimal weighting coefficients are the solution of the set of equations:

$$\begin{array}{ccccc} \mathrm{M}\xi_1\xi_\theta & = & \tilde{c}_1 \cdot \mathrm{M}\xi_1\xi_1 + & \tilde{c}_2 \cdot \mathrm{M}\xi_1\xi_2 + & \ldots & \tilde{c}_\mathrm{N} \cdot \mathrm{M}\xi_1\xi_\mathrm{N} \\ \mathrm{M}\xi_2\xi_\theta & = & \tilde{c}_1 \cdot \mathrm{M}\xi_2\xi_1 + & \tilde{c}_2 \cdot \mathrm{M}\xi_2\xi_2 + & \ldots & \tilde{c}_\mathrm{N} \cdot \mathrm{M}\xi_2\xi_\mathrm{N} \\ \vdots & = & \vdots & \vdots & \ldots & \vdots \\ \mathrm{M}\xi_\mathrm{N}\xi_\theta & = & \tilde{c}_1 \cdot \mathrm{M}\xi_\mathrm{N}\xi_1 + & \tilde{c}_2 \cdot \mathrm{M}\xi_\mathrm{N}\xi_2 + & \ldots & \tilde{c}_\mathrm{N} \cdot \mathrm{M}\xi_\mathrm{N}\xi_\mathrm{N} \end{array}$$

- Or using a shorter notation:

$$\mathbf{g} = \mathbf{G} \cdot \tilde{\mathbf{c}}$$

- Prediction error when using optimal coefficients:

$$\varepsilon(\tilde{\mathbf{c}}) = \mathbf{M}\left(\xi_\theta - \sum_{i=1}^{N} \tilde{c}_i \cdot \xi_i\right)^2 = \mathbf{M}\xi_\theta^2 - \mathbf{M}\xi_\theta \sum_{i=1}^{N} \tilde{c}_i \cdot \xi_i =$$

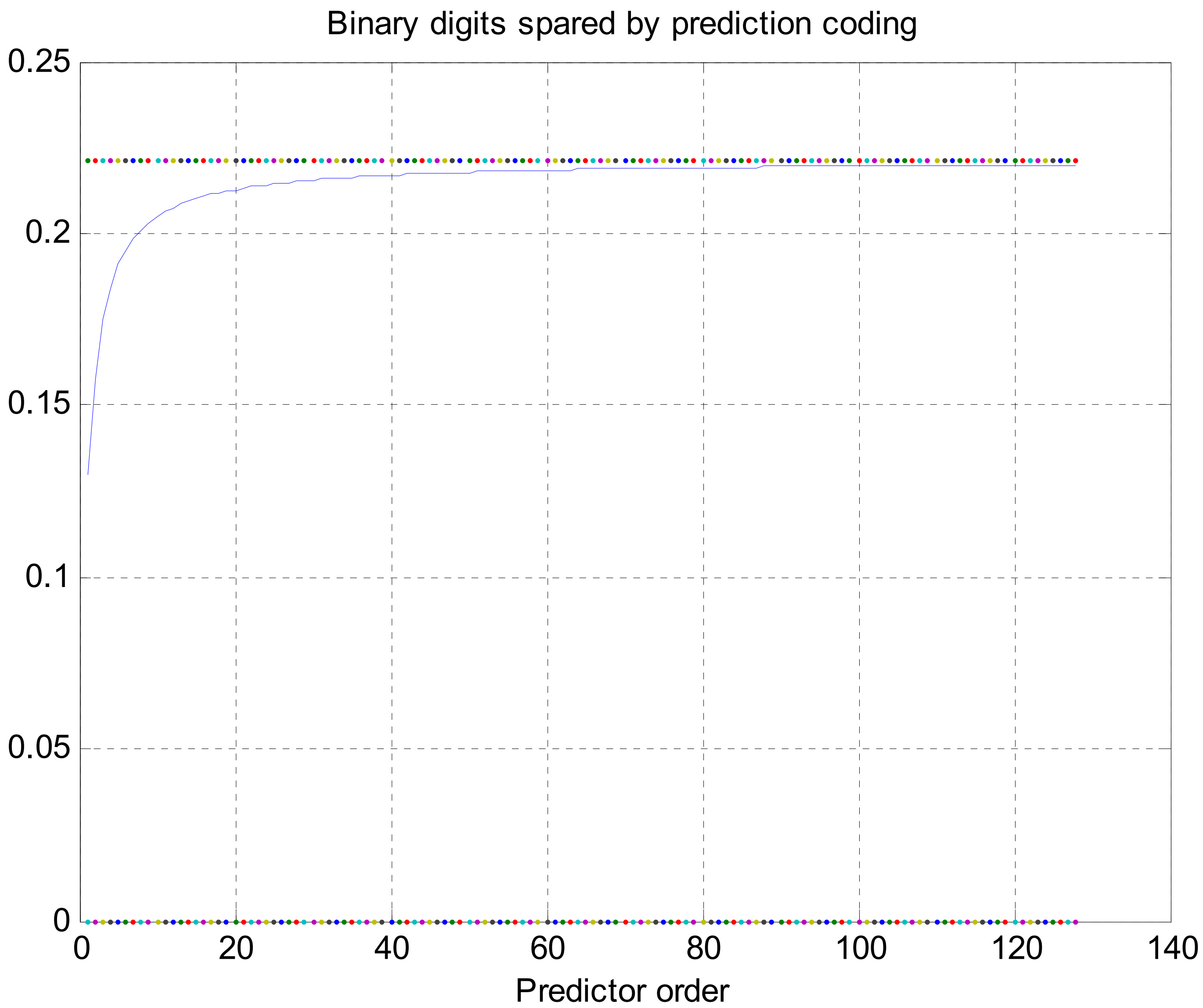
$$= \mathbf{M}\xi_\theta^2 - \sum_{i=1}^{N} \tilde{c}_i \cdot \mathbf{M}\xi_\theta \xi_i$$

- Prediction gain:

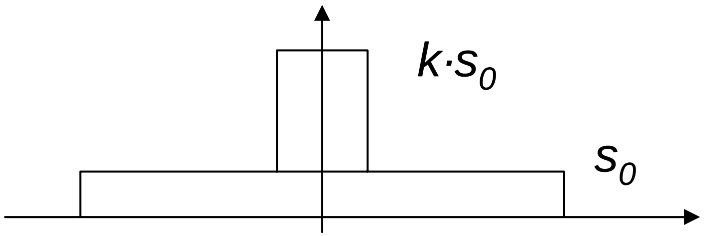
$$G_p = \frac{\mathbf{M}\xi_\theta^2}{\varepsilon(\tilde{\mathbf{c}})} = \frac{\mathbf{M}\xi_\theta^2}{\mathbf{M}\xi_\theta^2 - \sum_{i=1}^{N} \tilde{c}_i \cdot \mathbf{M}\xi_\theta \xi_i}$$

- and

$$\tilde{\mathbf{c}} = \mathbf{G}^{-1} \cdot \mathbf{g}$$


Binary digits spared by prediction coding
0.25
0.2
0.15
0.1
0.05
0
0
20
40
60
80
100
120
140
Predictor order

- Compression in the case of "dominant" spectrum



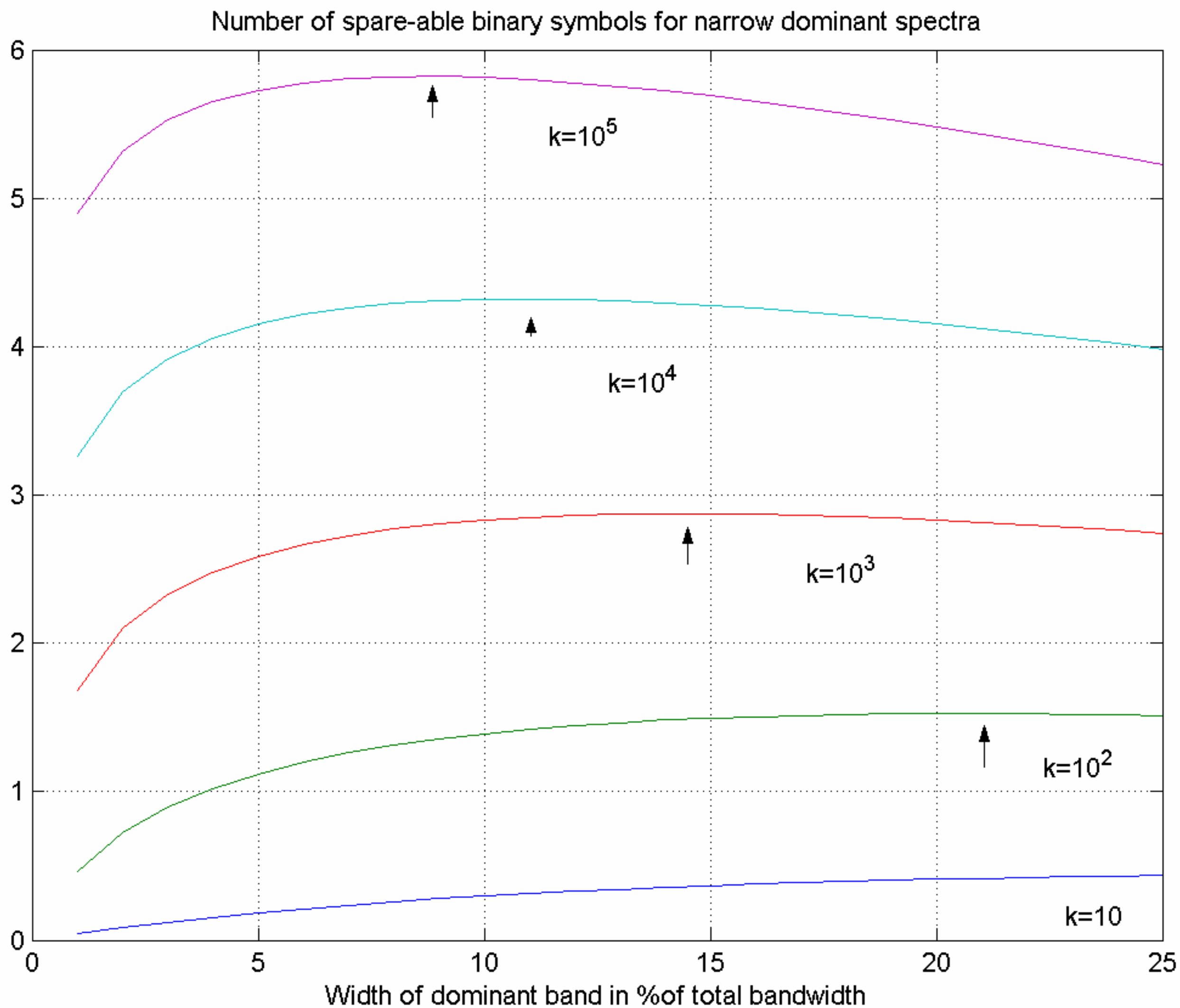
$$\bar{s}_{\xi} = \frac{2Bs_0 + 2b(k-1)s_0}{2B} =$$

$$= s_0[1 + \tfrac{b}{B}(k-1)]$$

$$\frac{1}{2}\log_2(G_p) = -\tfrac{1}{2}\cdot\tfrac{1}{2B}\int_{-B}^{B}\log_2\frac{s_{\xi}(\omega)}{\bar{s}_{\xi}}\,d\omega =$$

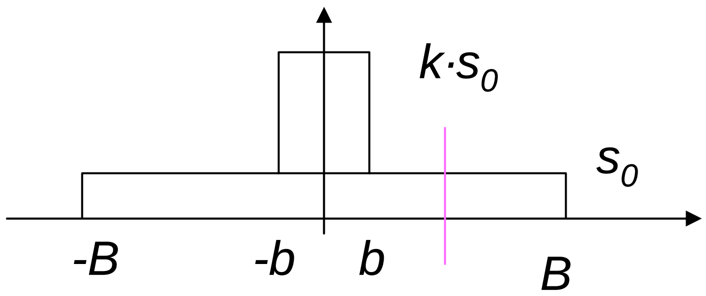
$$= -\frac{1}{2}\cdot\frac{2b\log_2\frac{k}{1+\frac{b}{B}(k-1)} + 2(B-b)\log_2\frac{1}{1+\frac{b}{B}(k-1)}}{2B} =$$

$$= -\tfrac{1}{2}\left(\tfrac{b}{B}\log_2\frac{k}{1+\frac{b}{B}(k-1)} + (1-\tfrac{b}{B})\log_2\frac{1}{1+\frac{b}{B}(k-1)}\right) = -\tfrac{1}{2}\cdot\log_2\frac{(k)^{\frac{b}{B}}}{1+\frac{b}{B}(k-1)}$$



Number of spare-able binary symbols for narrow dominant spectra
6
5
4
3
2
1
0
0
5
10
15
20
25
k=10^5
k=10^4
k=10^3
k=10^2
k=10
Width of dominant band in %of total bandwidth

- Sub-band coding



$$\frac{1}{2}\cdot\log_2\frac{\frac{1}{K}\sum_{i=1}^{K}\mathbf{M}\xi_i^2}{\left(\prod_{i=1}^{K}\mathbf{M}\xi_i^2\right)^{\frac{1}{K}}}$$

$$K=2; k=10; b=0.1B$$

$$\frac{1}{2}\cdot\log_2\frac{\frac{1}{K}\sum_{i=1}^{K}\mathbf{M}\xi_i^2}{\left(\prod_{i=1}^{K}\mathbf{M}\xi_i^2\right)^{\frac{1}{K}}}=\frac{1}{2}\cdot\log_2\frac{\frac{1}{2}\left(14bs_0+5bs_0\right)}{\sqrt{14bs_0\cdot 5bs_0}}=\frac{1}{2}\cdot\log_2\frac{9.5bs_0}{\sqrt{70}\cdot bs_0}=0.09$$

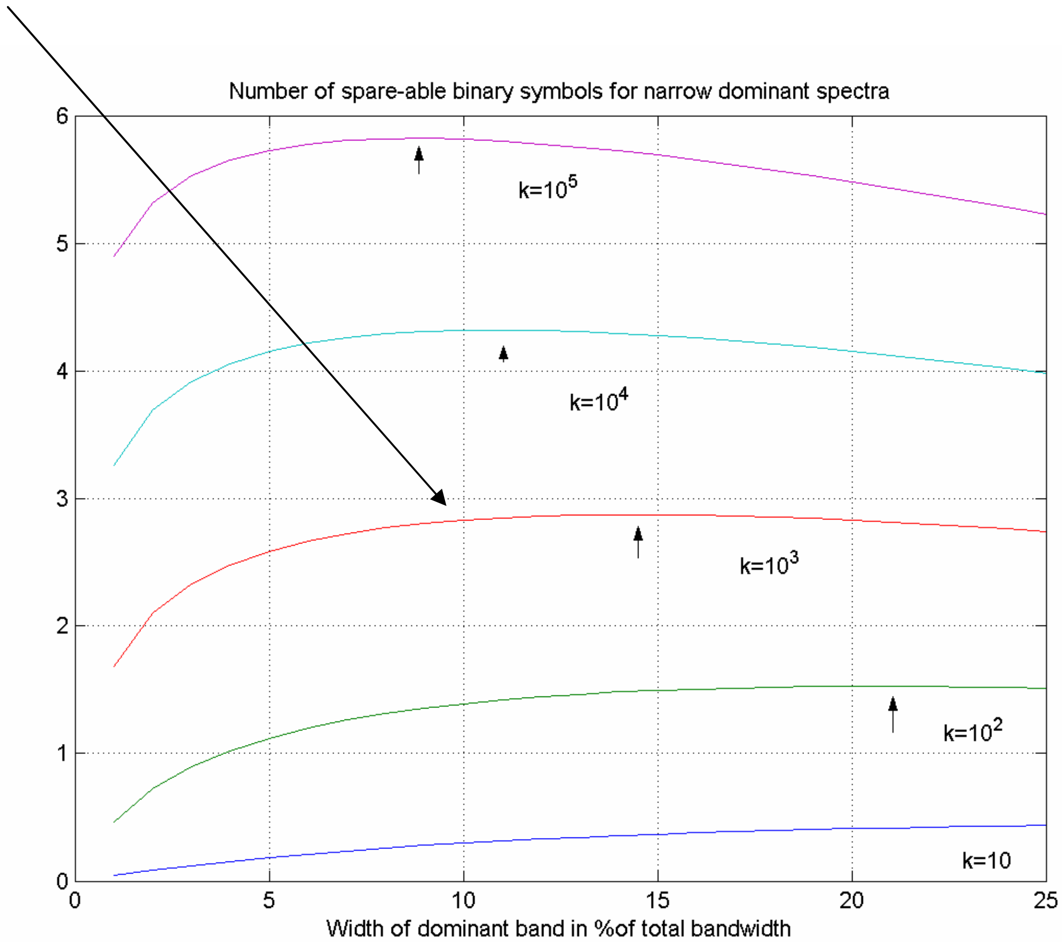
$$K=10; k=10; b=0.1B$$

$$\frac{1}{2}\cdot\log_2\frac{\frac{1}{10}\left(10bs_0+9bs_0\right)}{\sqrt[10]{10bs_0\cdot(bs_0)^9}}=\frac{1}{2}\cdot\log_2\frac{1.9bs_0}{\sqrt[10]{10}\cdot bs_0}=0.297$$

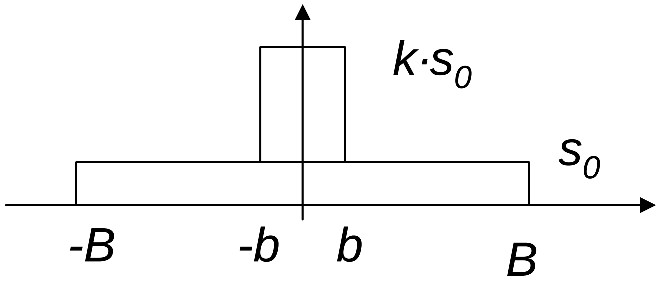
$$K = 10; k = 1000; b = 0.1B$$

$$\frac{1}{2} \cdot \log_2 \frac{\frac{1}{K}\left(kbs_0 + (K-1)bs_0\right)}{\sqrt[K]{kbs_0 \cdot (bs_0)^{K-1}}} = \frac{1}{2} \cdot \log_2 \frac{\frac{1}{10}\left(1000bs_0 + 9bs_0\right)}{\sqrt[10]{1000bs_0 \cdot (bs_0)^9}} =$$

$$= \frac{1}{2} \cdot \log_2 \frac{100.9bs_0}{\sqrt[10]{1000} \cdot bs_0} = 2.8301$$



- KL transformation of signal with dominant spectr.

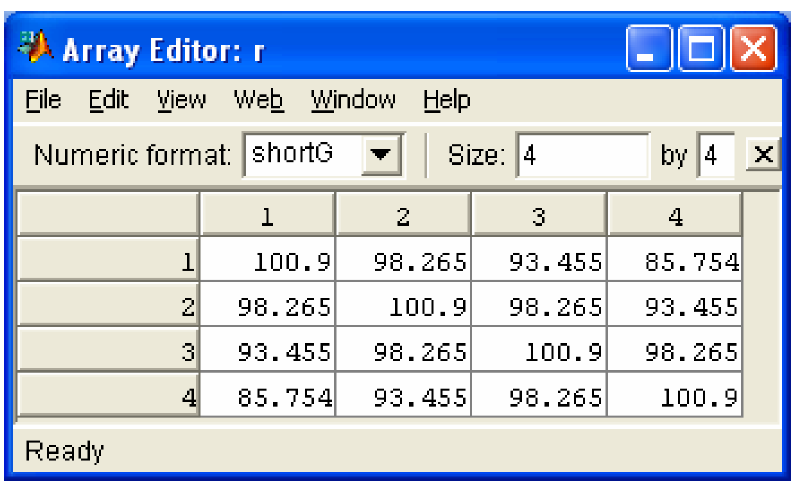


$$R_\xi(\tau) = 2Bs_0 \cdot \frac{\sin(B\tau)}{(B\tau)} + 2b(k-1)s_0 \cdot \frac{\sin(b\tau)}{(b\tau)}$$

$$\tau_0 = \pi / B;$$

$$R_\xi(n \cdot \tau_0) = 2Bs_0 \left[\mathrm{sinc}(\mathrm{n} \cdot \pi) + (k-1)\frac{b}{B}\mathrm{sinc}(\mathrm{n}\tfrac{\mathrm{b}}{\mathrm{B}}\pi) \right]$$

- Ex.: K=4; k=1000; b=0.1*B



	1	2	3	4
1	100.9	98.265	93.455	85.754
2	98.265	100.9	98.265	93.455
3	93.455	98.265	100.9	98.265
4	85.754	93.455	98.265	100.9

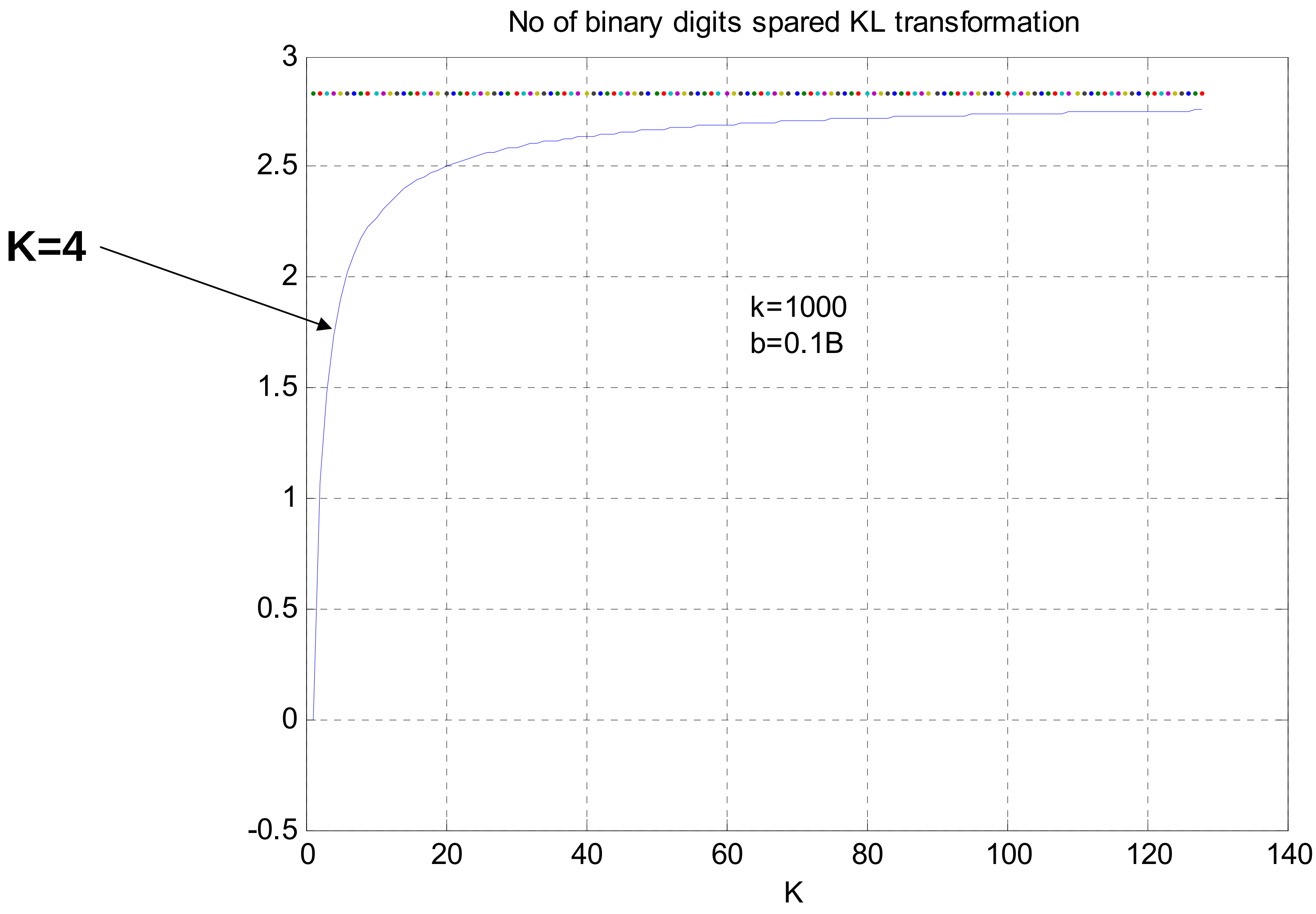
$$\mathbf{R}_\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.09 & 0 & 0 \\ 0 & 0 & 16.8 & 0 \\ 0 & 0 & 0 & 384 \end{pmatrix}$$

- Calculating when the # of elements in blocks is 4

$$\frac{1}{2}\cdot ld\,\frac{\frac{1}{K}\sum_{i=1}^{K}\lambda_i}{\left(\prod_{i=1}^{K}\lambda_i\right)^{1/K}} = \frac{1}{2}\cdot ld\,\frac{\frac{1}{4}(1+1.09+16.8+384)}{(1*1.09*16.8*384)^{1/4}} =$$

$$= \frac{1}{2}\cdot ld\,\frac{\frac{1}{4}\cdot 403.6}{(7018.6)^{1/4}} = 1.7313$$

- Next slide shows the result for longer blocks


No of binary digits spared KL transformation
K=4
k=1000
b=0.1B
3
2.5
2
1.5
1
0.5
0
-0.5
0
20
40
60
80
100
120
140
K