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The MM $\sum_{k=1}^{K}$ CPP$_k$/GE/c/L G-queue with heterogeneous servers: Steady state solution and an application to performance evaluation

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Abstract

A new queue, referred to here as *the HetSigma queue*, in the Markovian framework, is proposed in order to model nodes in modern telecommunication networks. The queue has many of the necessary ingredients, such as joint (or individual) Markov modulation of the arrival and service processes, superposition of K CPP (compound Poisson process) streams of (positive) customer arrivals, and a CPP of negative customer arrival stream in each of the modulating phases, a multiserver with c non-identical (can also be identical) servers, GE (generalised exponential) service times in each of the modulating phases and a buffer of finite or infinite capacity. Thus, the model can accommodate correlations of the inter-arrival times (of batches), and geometric as well as non-geometric batch size distributions of customers in both arrivals and services. The use of negative customers can facilitate modelling server failures, packet losses, load balancing, channel impairment in wireless networks, and in many other applications. An exact and computationally efficient solution of this new queue for steady state probabilities and performance measures is developed and presented.

A non-trivial application of the new queue to the performance evaluation of a wireless communication system is presented, along with numerical results, to illustrate the efficacy of the proposed method. The use of negative customers is also demonstrated. The new queue, perhaps with further evolution, has the potential to emerge as a *generalised Markovian node model*.

Keywords: MMCPP; Compound Poisson process; Generalised exponential; Unbounded batch-arrivals; Unbounded batch-services; Generalised Markovian node model; Spectral expansion; Mathematical transformations; Performance evaluation; Wireless networks

1. Introduction

Modelling packet traffic and nodes in modern communication networks has become complicated because of the existence of burstiness (time varying arrival or service rates, arrivals or services of packets in *batches*) and important correlations among inter-arrival times [1]. The MMPP (Markov Modulated Poisson Process) [2] was an attempt to model the correlations effectively. A number of important applications have used the MMPP in performance evaluation

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over the years [3–7]. Excellent literature also exists in model fitting and parameter estimation of the real traffic by MMPP [7–10]. However, the MMPP cannot accommodate batch-arrivals and/or batch-services.

The MM CPP/GE/c queue [11] and the MM CPP/GE/c/L G-queue [12] have emerged as effective models, in the Markovian framework, in order to accommodate inter-arrival time correlations, large or unbounded batch sizes and burstiness. However, this was mostly limited to geometric batch size distributions [13], with only very little flexibility towards non-geometric batch sizes. In these queues, the steady state balance equations are highly complicated and hard to solve. The method of *Successive Simultaneous weighted Subtractions* (SSS) was used on the steady state balance equations, transforming them to a computable form of the QBD-M (*Quasi simultaneous-bounded-multiple Births and simultaneous-bounded-multiple Deaths*) type and then solved very efficiently by the spectral expansion method [11,12]. The moments of the inter-batch departure interval and the departure burst size distribution are derived in [12] and the sojourn time distribution in [14], for the MM CPP/GE/c/L G-queue.

Recently, we have proposed and solved the MM $\sum_{k=}^{K}$ CPP$_k$/GE/c/L G-queue with homogeneous servers (*the Sigma queue*) [15,16], which is capable of modelling geometric as well as non-geometric batch size distributions, to an extent. The effective arrival process in this queue is the MM $\sum_{k=1}^{K}$ CPP$_k$ which is thus a much more useful generalisation of the CPP, MMPP and the MMCPP. A procedure consisting of two steps of transformations is used to transform the steady state balance equations to the QBD-M type computable form and efficient computational procedure has been developed for the steady state probabilities [15–17]. This queue has been used effectively for a non-trivial and novel application in the performance evaluation of optical communication nodes. In that work [17], negative customers were applied to model the packet-loss due to certain technology constraint, which is an additional contribution to the application of negative customers [18,19]. Negative customers were used in the past to model server-failures in computing and communication systems [20] and in manufacturing systems [21].

Queues with heterogeneous multiservers with possibly different characteristics do take into account many practical aspects in modelling real systems. Models for many practical scenarios (e.g., servers formed with different types of processors as a consequence of system updates, nodes in telecommunications network with links of different capacities, nodes in wireless systems serving different mobile users, etc.) involve heterogeneous servers. In [22], by the method of stochastic sample path comparisons, Lehtonen has proved that heterogeneous multiserver systems are superior in performance (this result can be extended to reliability as well) to the homogeneous ones, with the same total service time. This is also established by Stecke [23] in the context of manufacturing cells and systems, and very recently by Trancoso [24]. Trancoso's extensive simulation results in [24] do lead to the conclusion that, when configuring a multiprocessor system, it is necessary to consider seriously a heterogeneous configuration.

In this paper, we introduce and use a more powerful queueing model, the MM $\sum_{k=1}^{K}$ CPP$_k$/GE/c/L G-queue with heterogeneous servers (*the HetSigma model/queue*). Steady state balance equations of this model are further more complicated; therefore an efficient method is needed to obtain the steady state probabilities. We present a transformational procedure consisting of three steps, to transform the balance equations into the QBM-type computable form. We also present the resulting equations and expressions (which are of huge size), and the computational procedure into a compact and neat form in order to achieve computational ease and efficiency. We also demonstrate the use of the *HetSigma* queue to model a specific case in wireless communications.

The earlier published queueing models, the MM CPP/GE/c, the MM CPP/GE/c/L G-queues and the *Sigma* queue, are thus special cases of this new queue, which, with possible further modifications, is capable of emerging as a *generalised Markovian node model* for telecommunication applications.

The rest of this paper is organised as follows. Section 2 introduces the system/model. Section 3 develops the steady state balance equations of the new queue. Section 4 presents the three steps of transforming the steady state balance equations into an efficiently computable form. Section 5 describes the solution of the transformed balance equations by the spectral expansion method and also the solution for the case of infinite queueing capacity. Section 6 presents a non-trivial application of the *HetSigma* queue for the performance evaluation of a wireless communication system, along with numerical results. The importance of the proposed queue, the efficacy of the proposed methodology to find the steady state performance measures and the computational efficiency are illustrated. Conclusions are drawn in Section 7. The Appendices present some theoretical and some practical proofs related to the methodology.

2. Model description

The system/model is a multi-server queue. It is described below, by illustrating the various components and their interactions.

2.1. Modulation

Both the arrival and service processes are modulated by the same continuous time, irreducible Markov process, called X, with N states (or phases of modulation). Let Q be the generator matrix of this modulating process (X). The off-diagonal element, $Q(i,k) = q_{i,k}$ $(i \neq k)$, is the instantaneous transition rate from phase i to phase k, and the ith diagonal element $Q(i,i) = -q_i = -\sum_{l=1}^{i-1} q_{i,l} - \sum_{l=i+1}^{N} q_{i,l}$.

Then, $\mathbf{r} = (r_1, r_2, \ldots, r_N)$, the vector of equilibrium probabilities of the modulating phases, is uniquely determined by the equations

$$\mathbf{r}Q = 0; \quad \mathbf{r}\mathbf{e}_N = 1,$$

where $\mathbf{e}_N$ stands for the column vector with N elements, each of which is unity.

The assumption, that the arrival and service processes are modulated by the same continuous time, irreducible Markov process with N modulating phases, does not limit the usage of the model. That is because the case of different and independent modulating processes for the arrivals and the services can be traced back to our model: i.e., if the number of the phases in the modulation of the arrival process is N_a and that of the service process is N_s independently, then we can convert this case into our model by considering an appropriate joint modulating Markov process with $N = N_s N_a$ states.

2.2. Customer arrival process

The arrival process is inherently Markov modulated by X. The arrivals, in each of the modulating phase, are the superposition of K independent CPP [25] arrival streams of positive customers (referred to as customers hereafter) and an independent CPP of negative customers, the parameters of which can depend on the phase of modulation. The customers (positive customers) of the K different arrival streams are not distinguishable. Strictly during the modulating phase i, the parameters of the GE inter-arrival time distribution of the kth $(k = 1, 2, \ldots, K)$ customer arrival stream are $(\sigma_{i,k}, \theta_{i,k})$, and (ρ_i, δ_i) are those of the negative customer arrival stream. That is, the probability distribution function of inter-arrival times $(\tau_{i,k})$, strictly during phase i for the kth stream of customers, is governed by $\Pr(\tau_{i,k} = 0) = \theta_{i,k}$ and $\Pr(0 < \tau_{i,k} < t) = (1 - \theta_{i,k})(1 - \mathrm{e}^{-\sigma_{i,k} t})$. Similar two probabilities pertaining to the case of the negative customers, strictly during phase i, are δ_i and $(1 - \delta_i)(1 - \mathrm{e}^{-\rho_i t})$ respectively. Thus, all the $K + 1$ arrival point-processes, strictly during a given modulating phase, can be seen as batch-Poisson, with batches arriving at each point having geometric size distribution. Specifically, strictly during phase i, the probability that a batch is of size s is $(1 - \theta_{i,k})\theta_{i,k}^{s-1}$, for the kth stream of customers, and $(1 - \delta_i)\delta_i^{s-1}$ for the negative customers.

Now, looking at the $K + 1$ arrival processes *overall*, each of them is an MMCPP, but with joint-modulation. Also, the overall superposed arrival process of positive customers is an MM$\sum_{k=1}^{K}$ CPP$_k$ which is a much more useful generalisation of the CPP, MMPP and the MMCPP arrival processes.

Let $\sigma_{i,.}$, $\overline{\sigma_{i,.}}$ be the average arrival rate of customer-batches and customers respectively, strictly during phase i. Let σ, $\overline{\sigma}$ be the overall average arrival rate of batches and customers, respectively. Then,

$$\sigma_{i,.} = \sum_{k=1}^{K} \sigma_{i,k}; \quad \overline{\sigma_{i,.}} = \sum_{k=1}^{K} \frac{\sigma_{i,k}}{(1 - \theta_{i,k})}; \quad \sigma = \sum_{i=1}^{N} \sigma_{i,.} r_i; \quad \overline{\sigma} = \sum_{i=1}^{N} \overline{\sigma_{i,.}} r_i \tag{1}$$

2.3. Arrival batch size distribution

In a modulating phase, the arrivals are the superposition of many CPP's with phase-dependent parameters. Hence, the superposed arrivals of customers, strictly during phase i, are like bulk-Poisson ($M^{[x]}$) with arrival rate $\sigma_{i,.}$ and with a batch size distribution $\{\pi_{l/i}\}$, that is more general than mere geometric. The probability that this batch size is l (strictly during phase i) is given by

$$\pi_{l/i} = \sum_{k=1}^{K} \frac{\sigma_{i,k}}{\sigma_{i,.}} (1 - \theta_{i,k})\theta_{i,k}^{l-1} \tag{2}$$

The overall batch size distribution is then given approximately by $\pi_{l/.} = \sum_{i=1}^{N} r_i \pi_{l/i}$.

It is worth emphasising that our queue handles the case of the unbounded batch sizes in arrivals (also in services, as we shall see later), for which, to our limited knowledge, there have never been any implemented solutions with practical numerical results, only with the exception of the earlier works [11,12,15]. To our limited knowledge, until now for the BMAP or the QBD-M process based models, practical numerical results exist only when the batch sizes are limited, with computational complexity sharply increasing with the maximum batch size allowed.

2.4. The GE multi-server and the queueing capacity

The service facility has c heterogeneous servers in parallel. A number of scheduling policies can be thought of. Though, in principle, a number of scheduling policies can indeed be modelled by following our methodology, the one that we have adopted in this paper, for illustration and detailed study, is described below in the following three paragraphs. It is assumed that a set of service priority assignments exists (or given/chosen) which identifies each server by a unique service priority: 1 is the highest and c is the lowest. This set can be chosen from the $c!$ different possible ways. However, the impact of choosing service priorities can be very high on the performance measures, whose study is not in the scope of this paper.

Each server is then numbered, without loss of generality, by its own service priority. The GE-distributed service time parameters of the nth server ($n = 1, 2, \ldots, c$), in phase i, are denoted by $(\mu_{i,n}, \phi_{i,n})$.

L is the queueing capacity, in all phases, including the customers in service, if any. L can be finite or infinite. We assume, when the number of customers is j and the arriving batch size of customers is greater than $L - j$ (assuming finite L), then only $L - j$ customers are taken in and the rest are rejected.

The service discipline is FCFS (First Come First Scheduled for service) and each server serves at most one positive customer at any given time. Customers, on their completion of service, leave the system. When the number of customers in the system, j, (including those in service if any) is $\geq c$, then only c customers are served with the remaining $(j - c)$ waiting for service. When $j < c$, only the first j servers, (i.e., servers numbered $1, 2, \ldots, j$), are occupied and the rest are idle. This is made possible by what is known as customer switching. Thus, when server n becomes idle, an awaiting customer would be taken up for service. If there is no awaiting customer, then a customer that is being served by the lowest possible priority server (i.e., among servers $c, c-1, \ldots, n+1$) switches to server n. In such a switching, the service time is governed by either *resume or repeat with resampling*, thus preserving the Markovian property. The switching is instantaneous and the switching time is treated as being negligible. Negative customers neither wait in the queue, nor are served. Though this may not be a preferred scheduling discipline in a number of situations, a variety of scheduling disciplines can indeed be modelled using our methodology, with appropriate modifications (cf. Section 5.3).

In any modulating phase, the operation of the GE server is similar to that described for the CPP arrival processes above. However, the batch size associated with a service completion is bounded by one more than the number of customers waiting to commence service at the departure instant. When $c \leq j < L + 1$, the maximum batch size at a departure instant is $j - c + 1$, only one server being able to complete a service period at any one instant under the assumption of exponentially distributed batch-service times. Thus, here the probability that a departing batch is of size s is $\sum_{n=1}^{c} \frac{\mu_{i,n}(1-\phi_{i,n})\phi_{i,n}^{s-1}}{\mu_{i.}}$ for $1 \leq s \leq j - c$ and $\sum_{n=1}^{c} \frac{\mu_{i,n}\phi_{i,n}^{j-c}}{\mu_{i.}}$ for $s = j - c + 1$, where $\mu_{i.} = \sum_{n=1}^{c} \mu_{i,n}$. However, when $1 \leq j \leq c$, the departing batch has size 1 since each customer is already engaged by a server and there are then no customers waiting to commence service.

It is assumed that the first positive customer in a batch arriving at an instant when the queue length is less than c (so that at least one server is free) never skips service (has non-zero service time) [12]. However, even without this assumption the general methodology described in this paper is still applicable, with appropriate minor modifications.

2.5. Negative customer semantics

The uses of negative customers with appropriate killing discipline are many, viz. for facilitating flow control studies, load balancing studies, to model breakdowns and to model packet losses caused by the arrival of corrupted packets, as explained in [12].

A negative customer removes a positive customer in the queue, according to a specified *killing discipline*. We consider here a variant of the RCE killing discipline (removal of customers from the end of the queue), where the most recent positive arrival is removed, but which does *not* allow a customer actually in service to be removed: a negative customer that arrives when there are no positive customers waiting to start service has no effect. We may say that customers in service are immune to negative customers or that the service itself is *immune servicing*. Such a killing discipline is suitable for modelling of load balancing where work is transferred from overloaded queues but never work that is actually in progress.

When a batch of negative customers of size l $(1 \leq l < j - c)$ arrives, any l positive customers, that are not being served, are removed from the queue, leaving the remaining $j - l$ positive customers in the system. If $l \geq j - c \geq 1$, then $j - c$ positive customers are removed, leaving none waiting to commence service (queue length equal to c). If $j \leq c$, the negative arrivals have no effect.

$\overline{\rho_i}$, the average arrival rate of negative customers strictly during phase i, and $\overline{\rho}$, the overall average arrival rate of negative customers, are given by

$$\overline{\rho_i} = \frac{\rho_i}{1 - \delta_i}; \quad \overline{\rho} = \sum_{i=1}^{N} r_i \overline{\rho_i}. \tag{3}$$

2.6. Condition for stability

When L is finite, the system is ergodic since the representing Markov process is irreducible. Otherwise, i.e., when the queueing capacity is unbounded, the overall average departure rate increases with the queue length, and its maximum (the overall average departure rate when the queue length tends to ∞) can be determined as

$$\overline{\mu}_{L\to\infty} = \sum_{n=1}^{c} \sum_{i=1}^{N} \frac{r_i \mu_{i,n}}{1 - \phi_{i,n}}. \tag{4}$$

Hence, we conjecture that the necessary and sufficient condition for the existence of steady state probabilities is

$$\overline{\sigma} < \overline{\rho} + \overline{\mu}_{L\to\infty}. \tag{5}$$

A rigorous proof of the above condition may possibly be obtained by following the methodology in [26]; however, this is an item for further research only.

3. The steady state balance equations

The state of the system at any time t can be specified completely by two integer-valued random variables, $I(t)$ and $J(t)$. $I(t)$ varies from 1 to N (known as operative states), representing the phase of the modulating Markov chain, and $0 \leq J(t) < L + 1$ represents the number of positive customers in the system at time t, including any in service. The system is now modelled by a continuous time discrete state Markov process, $\overline{Y}$ (Y if L is infinite), on a rectangular lattice strip. Let $I(t)$, the operative state, vary in the horizontal direction and $J(t)$, the queue length or the level, in the vertical direction. We denote the steady state probabilities by $\{p_{i,j}\}$, where $p_{i,j} = \lim_{t\to\infty} \Pr(I(t) = i, J(t) = j)$, and let $\mathbf{v}_j = (p_{1,j}, \ldots, p_{N,j})$.

The process $\overline{Y}$ evolves due to the following instantaneous transition rates:

(a) $q_{i,k}$ – purely lateral transition rate – from state (i, j) to state (k, j), for all $j \geq 0$ and $1 \leq i, k \leq N$ $(i \neq k)$, caused by a phase transition in the modulating Markov process;
(b) $B_{i,j,j+s}$ – s-step upward transition rate – from state (i, j) to state $(i, j + s)$, $\forall i$, caused by a new batch arrival of size s of positive customers. For a given j, s can be seen as bounded when L is finite and unbounded when L is infinite;

(c) $C_{i,j,j-s}$ – s-step downward transition rate – from state (i, j) to state $(i, j-s)$, $(j-s \geq c+1)$, $\forall i$, caused by either a batch service completion of size s or a batch arrival of negative customers of size s;
(d) $C_{i,c+s,c}$ – s-step downward transition rate – from state $(i, c+s)$ to state (i, c), $\forall i$, caused by a batch arrival of negative customers of size $\geq s$ or a batch service completion of size s, $(1 \leq s \leq L-c)$;
(e) $C_{i,c-1+s,c-1}$ – s-step downward transition rate, from state $(i, c-1+s)$ to state $(i, c-1)$, $\forall i$, caused by a batch departure of size s $(1 \leq s \leq L-c+1)$;
(f) $C_{i,j+1,j}$ – 1-step downward transition rate, from state $(i, j+1)$ to state (i, j), $(c \geq 2;\ 0 \leq j \leq c-2)$, $\forall i$, caused by a single departure.

We obtain

$$B_{i,j-s,j} = \sum_{k=1}^{K}(1-\theta_{i,k})\theta_{i,k}^{s-1}\sigma_{i,k} \quad (\forall i;\ 0 \leq j-s \leq L-2;\ j-s < j < L);$$

$$B_{i,j,L} = \sum_{k=1}^{K}\sum_{s=L-j}^{\infty}(1-\theta_{i,k})\theta_{i,k}^{s-1}\sigma_{i,k} = \sum_{k=1}^{K}\theta_{i,k}^{L-j-1}\sigma_{i,k} \quad (\forall i;\ j \leq L-1);$$

$$\begin{aligned}
C_{i,j+s,j} &= \sum_{n=1}^{c}\mu_{i,n}(1-\phi_{i,n})\phi_{i,n}^{s-1} + (1-\delta_i)\delta_i^{s-1}\rho_i \quad (\forall i;\ c+1 \leq j \leq L-1;\ 1 \leq s \leq L-j)\\
&= \sum_{n=1}^{c}\mu_{i,n}(1-\phi_{i,n})\phi_{i,n}^{s-1} + \delta_i^{s-1}\rho_i \quad (\forall i;\ j = c;\ 1 \leq s \leq L-c)\\
&= \sum_{n=1}^{c}\phi_{i,n}^{s-1}\mu_{i,n} \quad (\forall i;\ j = c-1;\ 1 \leq s \leq L-c+1)\\
&= 0 \quad (\forall i;\ c \geq 2;\ 0 \leq j \leq c-2;\ s \geq 2)\\
&= \sum_{n=1}^{j+1}\mu_{i,n} \quad (\forall i;\ c \geq 2;\ 0 \leq j \leq c-2;\ s = 1).
\end{aligned}$$

Define

$$\begin{aligned}
B_{j-s,j} &= \mathrm{Diag}\left[B_{1,j-s,j}, B_{2,j-s,j}, \ldots, B_{N,j-s,j}\right] \quad (j-s < j \leq L);\\
B_s &= B_{j-s,j} \quad (j < L)\\
&= \mathrm{Diag}\left[\sum_{k=1}^{K}\sigma_{1,k}(1-\theta_{1,k})\theta_{1,k}^{s-1}, \ldots, \sum_{k=1}^{K}\sigma_{N,k}(1-\theta_{N,k})\theta_{N,k}^{s-1}\right];\\
\Sigma_k &= \mathrm{Diag}\left[\sigma_{1,k}, \sigma_{2,k}, \ldots, \sigma_{N,k}\right] \quad (k = 1, 2, \ldots, K);\\
\Theta_k &= \mathrm{Diag}\left[\theta_{1,k}, \theta_{2,k}, \ldots, \theta_{N,k}\right] \quad (k = 1, 2, \ldots, K);\\
\Sigma &= \sum_{k=1}^{K}\Sigma_k;\\
R &= \mathrm{Diag}\left[\rho_1, \rho_2, \ldots, \rho_N\right]; \qquad \Delta = \mathrm{Diag}\left[\delta_1, \delta_2, \ldots, \delta_N\right];\\
M_n &= \mathrm{Diag}\left[\mu_{1,n}, \mu_{2,n}, \ldots, \mu_{N,n}\right] \quad (n = 1, 2, \ldots, c);\\
\Phi_n &= \mathrm{Diag}\left[\phi_{1,n}, \phi_{2,n}, \ldots, \phi_{N,n}\right] \quad (n = 1, 2, \ldots, c);\\
C_j &= \sum_{n=1}^{j}M_n \quad (1 \leq j \leq c);\\
&= \sum_{n=1}^{c}M_n = C \quad (j \geq c);\\
C_{j+s,j} &= \mathrm{Diag}\left[C_{1,j+s,j}, C_{2,j+s,j}, \ldots, C_{N,j+s,j}\right];\\
E &= \mathrm{Diag}(\mathbf{e}'_N)\ .
\end{aligned}$$

Then, we get

$$B_s = \sum_{k=1}^{K} \Theta_k^{s-1}(E - \Theta_k)\Sigma_k; \qquad B_1 = B = \sum_{k=1}^{K}(E - \Theta_k)\Sigma_k;$$

$$B_{L-s,L} = \sum_{k=1}^{K} \Theta_k^{s-1}\Sigma_k;$$

$$\begin{aligned} C_{j+s,j} &= \sum_{n=1}^{c} M_n(E - \Phi_n)\Phi_n^{s-1} + R(E - \Delta)\Delta^{s-1} \quad (c+1 \le j \le L-1;\ s = 1, 2, \ldots, L-j); \\ &= \sum_{n=1}^{c} M_n(E - \Phi_n)\Phi_n^{s-1} + R\Delta^{s-1} \quad (j = c;\ s = 1, 2, \ldots, L-c); \\ &= \sum_{n=1}^{c} M_n \Phi_n^{s-1} \quad (j = c-1;\ s = 1, 2, \ldots, L-c+1); \\ &= 0 \quad (c \ge 2;\ 0 \le j \le c-2;\ s \ge 2); \\ &= C_{j+1} \quad (c \ge 2;\ 0 \le j \le c-2;\ s = 1). \end{aligned}$$

The steady state balance equations are

(1) For the Lth row or level:

$$\sum_{s=1}^{L} \mathbf{v}_{L-s} B_{L-s,L} + \mathbf{v}_L [Q - C - R] = 0; \tag{6}$$

(2) For the jth row or level:

$$\sum_{s=1}^{j} \mathbf{v}_{j-s} B_s + \mathbf{v}_j \left[Q - \Sigma - C_j - RI_{j>c}\right] + \sum_{s=1}^{L-j} \mathbf{v}_{j+s} C_{j+s,j} = 0 \quad (0 \le j \le L-1); \tag{7}$$

(3) Normalisation

$$\sum_{j=0}^{L} \mathbf{v}_j \mathbf{e}_N = 1; \tag{8}$$

where $I_{j>c} = 1$ if $j > c$ else 0, and $\mathbf{e}_N$ is a column vector of size N with all ones.

In Y or $\overline{Y}$, the transitions from a level to any other level are possible, thus unbounded. These are, therefore, neither QBD nor QBD-M type. We call them QBD-U (*Quasi simultaneous-unbounded-multiple Births and simultaneous-unbounded-multiple Deaths*) processes.

For example, when $L = \infty$, it can be observed that (6) and (7) are an infinite number of equations in an infinite number of unknowns, viz. $v_0, v_1, \ldots$. Also, each of the balance equation is infinitely long containing all the infinite number of unknowns, viz. $v_0, v_1, \ldots$. The coefficient matrices of the unknown vectors are j-dependent. Therefore, these original balance equations actually do not have QBD or QBD-M structure. Hence, either the spectral expansion method or the other methods cannot be used directly to solve them.

In the next section we *transform* this system of equations to a QBD-M type *computable* form.

4. Transforming the balance equations

The method presented in this section is for sufficiently large L such that $L \ge 2c + K + 3$. When $L < 2c + K + 3$, then the Markov process $\overline{Y}$ can be solved by traditional methods [27].

Let us consider Eqs. (6)–(8). Each equation has *all* the unknown vectors $\mathbf{v_j}$'s. If L is unbounded, then these are infinite number of equations in infinite number of unknowns, $\mathbf{v_j}$'s, and each equation is infinitely long containing all the infinite number of unknowns. Also, the coefficient matrices of $\mathbf{v_j}$ are j-dependent. It may be noted that there has been neither a solution nor a solution methodology to solve these equations. In this paper a novel methodology is developed to solve these equations *exactly and efficiently*. First these complicated equations are *transformed* to a computable form by using certain transformations. The resulting transformed equations are of the QBD-M type and hence can be solved by one of the several available methods, viz. the spectral expansion method, the Bini–Meini method [31] or the matrix–geometric method with folding or block size enlargement [28].

Let the balance equations for level j be denoted by $\langle \mathbf{j} \rangle$. Hence, $\langle \mathbf{0} \rangle, \langle \mathbf{1} \rangle, \ldots, \langle \mathbf{j} \rangle, \ldots, \langle \mathbf{L} \rangle$ are the balance equations for the levels $0, 1, \ldots, j, \ldots, L$ respectively. Substituting $B_{L-s,L} = \sum_{k=1}^{K} \Theta_k^{s-1} \Sigma_k$ and $B_s = \sum_{k=1}^{K} \Theta_{.k}^{s-1}(E - \Theta_k)\Sigma_k$ in (6), (7), we get the balance equations for level L and for all the other levels as

$$\langle \mathbf{L} \rangle : \sum_{s=1}^{L} \sum_{k=1}^{K} \mathbf{v}_{L-s} \Theta_k^{s-1} \Sigma_k + \mathbf{v}_L [Q - C - R] = 0 \tag{9}$$

$$\langle \mathbf{L-1} \rangle : \sum_{s=1}^{L-1} \sum_{k=1}^{K} \mathbf{v}_{L-1-s} \Theta_k^{s-1} (E - \Theta_k)\Sigma_k + \mathbf{v}_{L-1} \left[Q - \Sigma - C_{L-1} - R\right] + \mathbf{v}_L C_{L,L-1} = 0 \tag{10}$$

$$\vdots$$

$$\langle \mathbf{j} \rangle : \sum_{s=1}^{j} \sum_{k=1}^{K} \mathbf{v}_{j-s} \Theta_k^{s-1} (E - \Theta_k)\Sigma_k + \mathbf{v}_j \left[Q - \Sigma - C_j - R\right] + \sum_{s=1}^{L-j} \mathbf{v}_{j+s} C_{j+s,j} = 0 \quad (j = L-2, L-3, \ldots, c+K+2) \tag{11}$$

$$\langle \mathbf{c+K+1} \rangle : \sum_{s=1}^{c+K+1} \sum_{k=1}^{K} \mathbf{v}_{c+K+1-s} \Theta_k^{s-1} (E - \Theta_k)\Sigma_k + \mathbf{v}_{c+K+1}[Q - \Sigma - C_{c+K+1} - R] + \sum_{s=1}^{L-c-K-1} \mathbf{v}_{c+K+1+s} C_{c+K+1+s,c+K+1} = 0 \tag{12}$$

$$\vdots$$

$$\langle \mathbf{j} \rangle : \sum_{s=1}^{j} \sum_{k=1}^{K} \mathbf{v}_{j-s} \Theta_k^{s-1} (E - \Theta_k)\Sigma_k + \mathbf{v}_j \left[Q - \Sigma - C_j - R I_{j>c}\right] + \sum_{s=1}^{L-j} \mathbf{v}_{j+s} C_{j+s,j} = 0 \quad (j = c+K, c+K-1, \ldots, 0). \tag{13}$$

Define the functions, $F_{K,l}$ $(l = 1, 2, \ldots, K)$ and $H_{c,n}$ $(n = 1, 2, \ldots, c)$ as

$$\begin{aligned} F_{K,l} &= \sum_{1 \le k_1 < k_2 < \ldots < k_l \le K} \Theta_{k_1} \Theta_{k_2} \ldots \Theta_{k_l} \quad (l = 1, 2, \ldots, K) \\ &= E \quad \text{if } l = 0 \\ &= 0 \quad \text{if } l \le -1 \text{ or } l > K \end{aligned} \tag{14}$$

$$\begin{aligned} H_{c,n} &= \sum_{1 \le k_1 < k_2 < \ldots < k_n \le c} \Phi_{k_1} \Phi_{k_2} \ldots \Phi_{k_n} \quad (n = 1, 2, \ldots, c) \\ &= E \quad \text{if } n = 0 \\ &= 0 \quad \text{if } n \le -1 \text{ or } n > c \end{aligned} \tag{15}$$

These functions have the following alternate definitions, properties and recursion by which they can be conceived and computed quite easily.

$$F_{k,0} = E, \quad F_{k,k} = \prod_{i=1}^{k} \Theta_i \quad (k = 1, 2, \ldots, K);$$

$$F_{k,l} = 0 \quad (k = 1, 2, \ldots, K; l < 0); \qquad F_{k,l} = 0 \quad (k = 1, 2, \ldots, K; l > k) \tag{16}$$

$$H_{m,0} = E, \quad H_{m,m} = \prod_{i=1}^{m} \Phi_i \quad (m = 1, 2, \ldots, c);$$
$$H_{m,n} = 0 \quad (m = 1, 2, \ldots, c; n < 0);$$
$$H_{m,n} = 0 \quad (m = 1, 2, \ldots, c;\ n > m). \tag{17}$$

The recursion, then, is

$$F_{1,0} = E; \quad F_{1,1} = \Theta_1;$$
$$F_{k,l} = F_{k-1,l} + \Theta_k F_{k-1,l-1} \quad (2 \le k \le K, 1 \le l \le k-1); \tag{18}$$
$$H_{1,0} = E; \quad H_{1,1} = \Phi_1;$$
$$H_{m,n} = H_{m-1,n} + \Phi_m H_{m-1,n-1} \quad (2 \le m \le c, 1 \le n \le m-1). \tag{19}$$

Transformation 1. *Modify simultaneously the balance equations for levels* $j\ (L-2-c \ge j \ge c+K+1)$, *by the transformation*

$$\langle \mathbf{j} \rangle^{(\mathbf{1})} \longleftarrow \langle \mathbf{j} \rangle + \sum_{l=1}^{K} (-1)^l \langle \mathbf{j} - \mathbf{l} \rangle F_{K,l} \quad (c+K+1 \le j \le L-2-c)$$
$$\langle \mathbf{j} \rangle^{(\mathbf{1})} \longleftarrow \langle \mathbf{j} \rangle \quad (j > L-2-c \text{ or } j < c+K+1).$$

Transformation 1 essentially replaces the balance equation for row j by a weighted linear sum of the balance equations of rows $j, j-1, \ldots, j-K$. This is done simultaneously to rows $(c+K+1 \le j \le L-2-c)$.

Apply the second transformation to the resulting equations.

Transformation 2. *Modify simultaneously the balance equations for levels* $j\ (L-2-c \ge j \ge c+K+1)$, *by the transformation*

$$\langle \mathbf{j} \rangle^{(\mathbf{2})} \longleftarrow \langle \mathbf{j} \rangle^{(\mathbf{1})} + \sum_{n=1}^{c} (-1)^n \langle \mathbf{j} + \mathbf{n} \rangle^{(\mathbf{1})} H_{c,n} \quad (c+K+1 \le j \le L-2-c)$$
$$\langle \mathbf{j} \rangle^{(\mathbf{2})} \longleftarrow \langle \mathbf{j} \rangle^{(\mathbf{1})} \quad (j > L-2-c \text{ or } j < c+K+1).$$

Apply the third and final transformation to the resulting equations.

Transformation 3. *Modify simultaneously the balance equations for levels* $j\ (L-2-c \ge j \ge c+K+1)$, *by the transformation*

$$\langle \mathbf{j} \rangle^{(\mathbf{3})} \longleftarrow \langle \mathbf{j} \rangle^{(\mathbf{2})} - \langle \mathbf{j} + \mathbf{1} \rangle^{(\mathbf{2})} \Delta \quad (c+K+1 \le j \le L-2-c)$$
$$\langle \mathbf{j} \rangle^{(\mathbf{3})} \longleftarrow \langle \mathbf{j} \rangle^{(\mathbf{2})} \quad (j > L-2-c \text{ or } j < c+K+1).$$

With these above three transformations, the transformed balance equation, $\langle \mathbf{j} \rangle^{(\mathbf{3})}$'s, for the rows $(c+K+1 \le j \le L-2-c)$, will be of the form

$$\mathbf{v}_{j-K} Q_0 + \mathbf{v}_{j-K+1} Q_1 + \cdots + \mathbf{v}_{j+c+1} Q_{K+c+1} = 0 \quad (j = L-2-c, L-1-c, \ldots, c+K+1) \tag{20}$$

where $Q_0, Q_1, \ldots, Q_{K+c+1}$ are $K+c+2$ number of j-independent matrices which can be easily derived algebraically from the system parameters by following the above-mentioned transformation procedures. The computational procedures and the theoretical as well as the practical proofs concerning this are dealt with, in Appendix, separately.

Thus, the resulting Eq. (20) corresponding to the rows from $c+K+1$ to $L-2-c$ are of the same form as those of the QBD-M processes [16] and hence have an efficient solution by several methods such as the spectral expansion method [29,30], the Bini–Meini method [31], the matrix–geometric method with folding or block size enlargement [28,32,33].

5. Spectral expansion solution of the balance equations

The set of Eq. (20) concerning the levels $c+K+1$ to $L-2-c$ have the coefficient matrices $Q_0, Q_1, \ldots, Q_{K+c+1}$ that are independent of j and hence have an efficient solution by the spectral expansion method [30,29,34]. These Q_l's ($l = 0, 1, \ldots, K+c+1$) can be obtained quite easily following the computational procedure in Appendix A.

Define the matrix polynomials $Z(\lambda)$ and $\overline{Z}(\xi)$ as

$$Z(\lambda) = Q_0 + Q_1\lambda + Q_2\lambda^2 + \cdots + Q_{K+c+1}\lambda^{K+c+1}, \tag{21}$$

$$\overline{Z}(\xi) = Q_{K+c+1} + Q_{K+c}\xi + Q_{K+c-1}\xi^2 + \cdots + Q_0\xi^{K+c+1}. \tag{22}$$

Then, the spectral expansion solution for $\mathbf{v}_j$ $(c+1 \le j \le L-1)$ is given by

$$\mathbf{v}_j = \sum_{l=1}^{KN} a_l \boldsymbol{\psi}_l \lambda_l^{j-c-1} + \sum_{l=1}^{(c+1)N} b_l \boldsymbol{\gamma}_l \xi_l^{L-1-j} \quad (c+1 \le j \le L-1) \tag{23}$$

where λ_l $(l = 1, 2, \ldots, KN)$ are the KN eigenvalues of least absolute value of the matrix polynomial $Z(\lambda)$ and ξ_l $(l = 1, 2, \ldots, (c+1)N)$ are the $(c+1)N$ eigenvalues of least absolute value of the matrix polynomial $\overline{Z}(\xi)$. $\boldsymbol{\psi}_l$ and $\boldsymbol{\gamma}_l$ are the left-eigenvectors of $Z(\lambda)$ and $\overline{Z}(\xi)$ respectively, corresponding to the eigenvalues λ_l and ξ_l respectively. a_l's and b_l's are arbitrary constants to be determined later.

It is shown in Appendix B that the matrix $\sum_{l=0}^{K+c+1} Q_l$ is singular, so $\lambda = 1$ is an eigenvalue on the unit-circle for both $Z(\lambda)$ and $\overline{Z}(\xi)$. If (5) is satisfied, the number of eigenvalues of $Z(\lambda)$ that are strictly within the unit circle is KN. If (5) is not satisfied, that number is $KN - 1$. These and also certain other properties of these eigenvalues, eigenvectors, also the relevant spectral analysis are dealt with (some of them are proved, others explained in detail) in [29,30]. Some of them are summarised below. Let the rank of Q_0 be $N - n_0$ and that of Q_{K+c+1} be $N - n_{K+c+1}$.

Then,

(a) $Z(\lambda)$ would have $d = (K+c+1)N - n_{K+c+1}$ eigenvalues of which n_0 are zero eigenvalues (also referred to as null eigenvalues), whereas $\overline{Z}(\xi)$ would have n_{K+c+1} zero eigenvalues and $(K+c+1)N - n_0 - n_{K+c+1}$ non-zero eigenvalues.
(b) If $(\lambda \ne 0, \boldsymbol{\psi})$ is a non-zero eigenvalue-eigenvector pair of $Z(\lambda)$, then there exists a corresponding non-zero eigenvalue-eigenvector pair, $(\xi = \frac{1}{\lambda}, \boldsymbol{\gamma} = \boldsymbol{\psi})$ for $\overline{Z}(\xi)$. Thus, the non-zero eigenvalues of these two matrix polynomials are mutually reciprocal.
(c) The KN eigenvalues of least absolute value of $Z(\lambda)$ and the $(c+1)N$ eigenvalues of least absolute value of $\overline{Z}(\xi)$ lie either strictly inside, or on, their respective unit-circles, but not outside.
(d) There is no problem posed by multiple eigenvalues, i.e., independent eigenvectors having coincident eigenvalues, since each pair $(\boldsymbol{\lambda}, \boldsymbol{\psi})$ is distinct.

If the unknowns a_l's and b_l's are determined in such a way that all the balance equations are satisfied, then the vectors $\mathbf{v}_j$ $(c+1 \le j \le L-1)$ can be computed from the steady state solution (23). Hence, the unknowns are the scalars, $a_1, a_2, \ldots, a_{K\cdot N}, b_1, b_2, \ldots, b_{(c+1)N}$, and the vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_c, \mathbf{v}_L$. These are totally $KN + (c+1)N + (c+2)N = (2c+K+3)N$ scalar unknowns. In order to solve for them, we still have the transformed balance equations concerning the levels $0, 1, \ldots, c+K, L-1-c, L-c, \ldots, L$ and also Eq. (8). These are $(c+K+1+c+2)N+1$ linear simultaneous equations in the above $(2c+K+3)N$ scalar unknowns. Of these equations only $(2c+K+3)N$ equations (including Eq. (8)) are independent. Hence, all these unknowns can be solved for. However, a *substantial* simplification of this task of finding the unknowns, leading to a substantial reduction to the number of equations to be solved, can be achieved by following the procedures in [29] given for similar situations.

5.1. System with infinite queueing capacity

So far the analysis has been for the case of finite L. A corresponding analysis for the case of infinite queueing capacity, when the stability condition is satisfied, yields the solution

$$\mathbf{v}_j = \sum_{l=1}^{KN} a_l \boldsymbol{\psi}_l \lambda_l^{j-c} \quad (j = c+1, c+2, \ldots). \tag{24}$$

Here, we need only the KN relevant eigenvalues-eigenvectors of $Z(\lambda)$ and the KN a_k's (which exist as real or complex conjugate pairs) are to be computed. Notice that Eq. (24) is the same as (23) when the limit $L \to \infty$ is taken. Also notice that the computation time for this case would be much less than that for finite L.

5.2. The case of c identical servers

In this case the flexibility to accommodate given non-geometric batch size distributions in services is reduced. However, the transformations can be simpler and the computation can be more efficient, as shown in [16].

5.3. No customer switching between servers

An efficient solution is possible for this case too, in which switching of customers from one server to another is prohibited. Here, when $j = 0$ or $c \le j < L + 1$, the representation of the states of the system is just as before, that is (i, j) $(i = 1, 2, \ldots, N)$. However, when $1 < j < c$, the operative state should represent both the modulating phase and also the state of servers-occupation. The number of servers-occupation states is $\binom{c}{j}$, which is the number of combinations of choosing j objects from c distinguishable objects. Hence, the number of operative states is $N\binom{c}{j}$. Also, it is to be noted that, when $j < c - 1$, there is more than one server available for the arriving customer and hence in such situations a scheduling policy has to be defined.

In this case, too, we are confident that appropriate transformations and a spectral expansion solution can be worked out for $\mathbf{v}_j$ $(j \ge c + 1)$, and hence an exact and efficient computational solution. However, this is beyond the scope of this paper, and remains an open problem for further research.

6. Application, numerical study and validation

In this section we present a brief performance study of wireless channels based on the application of the *HetSigma* queue. The aims of this study are to demonstrate (i) that the *HetSigma* queue can be used effectively to model certain situations in telecommunication systems involving several heterogeneous serving facilities, (ii) the correctness of the algorithm by comparing the numerical results with those obtained by directly solving of all the original balance Eqs. (6) and (7) along with the normalisation Eq. (8) (*direct solution method*), and (iii) the viability, applicability and the computational efficiency of the proposed method. Note that a much more detailed study of the wireless communication system presented below is beyond the scope of this paper, and may be pursued as an important item for further research.

6.1. Wireless communication scenario

In wireless broadband networks (for example, IEEE 802.11 WiFi, IEEE 802.16 WiMaX), APs (Access Points) are applied to control the channel assignment (in frequency and in time) and distribute traffic to mobile stations. APs may apply different adaptive mechanisms such as adaptive modulation and channel coding. The wireless channel quality is characterised by the received signal-to-noise ratio (SNR). As a result, before the transmission of the concrete physical symbol frame, the wireless channel is assigned with a burst profile (adaptive modulation and coding[1]) depending on the current received SNR.

We consider a scenario where an AP serves two wireless stations through two channels in the same frequency (that is, TDMA, Time Division Multiple Access, is used). Each wireless station initiates one traffic session (for example, download from servers in the Internet). Note that these two channels are shared by two wireless stations (for example, when there are waiting packets to be transferred to one station, the AP may utilise both timesharing slots to send packets to that station). Since the channels are in the same frequency, the fading behavior of the two channels can be assumed to be the same.

[1] Packets arriving from several flows are stored in the buffer of the AP, they are then coded and modulated before getting embedded into physical layer symbol frames and transmitted.

Table 1
Parameters used for the numerical study

$N = 5$, $K = 2$, $c = 2$
$$Q = \begin{bmatrix} -6.12443 & 4.11368 & 1.27748 & 0.530042 & 0.203223 \\ 4.11368 & -10.7775 & 5.10993 & 1.1926 & 0.361286 \\ 1.27748 & 5.10993 & -11.9707 & 4.77038 & 0.812893 \\ 0.530042 & 1.1926 & 4.77038 & -9.74459 & 3.25157 \\ 0.203223 & 0.361286 & 0.812893 & 3.25157 & -4.62897 \end{bmatrix}$$
$$[\sigma_{i,k}]_{5\times 2} = \begin{bmatrix} 215.23 & 286.70 \\ 215.23 & 286.70 \\ 215.23 & 286.70 \\ 215.23 & 286.70 \\ 215.23 & 286.70 \end{bmatrix}, [\theta_{i,k}]_{5\times 2} = \begin{bmatrix} 0.59332 & 0.06286 \\ 0.59332 & 0.06286 \\ 0.59332 & 0.06286 \\ 0.59332 & 0.06286 \\ 0.59332 & 0.06286 \end{bmatrix}, [\mu_{i,k}]_{5\times 2} = \begin{bmatrix} 235.78 & 176.79 \\ 353.68 & 265.18 \\ 424.41 & 318.22 \\ 565.88 & 424.28 \\ 707.36 & 530.36 \end{bmatrix}$$
$\phi_{i,k} = 0.12\ (\forall i, k)$, $\rho_1 = 30$, $\rho_2 = 50$, $\rho_3 = 10$, $\rho_4 = 20$, $\rho_5 = 10$, $\delta_i = 0.1$

6.2. Performance model

We assume that the degradations of a wireless channel can be described by Nakagami-m fading channel. That is, the SNR follows the Nakagami distribution [35]. We use the approach in [36] to divide the range of the SNR into five intervals and use a continuous time first-order Markov chain of five states to characterise fading channels. We model traffic generated by two sessions as two heterogeneous CPPs (i.e., $K = 2$). The generated packets are independent from the channel conditions and are to be transmitted by two channels which correspond to two heterogeneous servers ($c = 2$) in our case. This model is quite appropriate considering switching from server 2 to server 1 takes place rather very rarely, under the used load conditions, and it almost amounts to the case of “no customer-switching”. As a consequence, the proposed queueing model based on the *HetSigma* queue is inherently modulated by the Markov chain characterising the fading channels. That is, the modulating process has five states ($N = 5$). Based on the method proposed in [37] the infinitesimal matrix Q can be determined.

There may be some cases when packets (being transmitted) are lost due to the Quality of Service (QoS) degradation of the wireless channels (in such cases the wireless stations cannot reconstruct packets correctly despite the fact that it applies the advanced coding and modulation algorithm). To account for such packet losses, one can use an appropriate negative customer arrival stream.

The parameters of the customer CPPs and those of the GE service times, in different phases, can be easily obtained from the traces, using the first two moments of inter-arrival times, as was done in [15]. The CPP parameters of the negative customer stream, just required to account for the packet-losses, can also be obtained from the trace describing the channel behaviors. Note that the parameters of the GE distribution are determined by the first two moments of the samples from the realisation of the stochastic process described by the GE process [25,38]. The GE distribution is, in fact, the only distribution that is of least bias [25], if only the mean and variance are reliably computed from the samples. Since the study of the wireless system with real traces is not in the scope of the present paper, we have chosen the numerical values of the parameters ($\sigma_{i,k}, \theta_{i,k}, \rho_i, \delta_i, \mu_{i,n}, \phi_{i,n}$) of the model, rather artificially, only in order to demonstrate the applicability of our method to the case explained above, along with its efficacy, computational efficiency, correctness and usefulness. Once these are demonstrated, the methodology can easily be applied to real traces in order to get practical solutions in real case studies.

The parameter values used in the numerical study are given in Table 1. For numerical results, we have implemented the proposed method in *Mathematica* (http://www.wolfram.com). The program, containing a procedure for the steady state solution of this system by on our method (i.e., transformations + spectral expansion), has a *Mathematica* source code of approximately 300 lines. The results are obtained on a Fujitsu-Siemens T4010 notebook with 1.8 GHz Intel Centrino processor and 1 Gbyte memory.

The numerical results obtained by our approach (transformations + spectral expansion) are in excellent agreement with those obtained by the direct solution method.

In Fig. 1 we plot several curves of the runtimes (on log-scale) versus L (the queueing capacity of the system on log-scale). From the results one can make the following observations concerning the computational efficiency of the

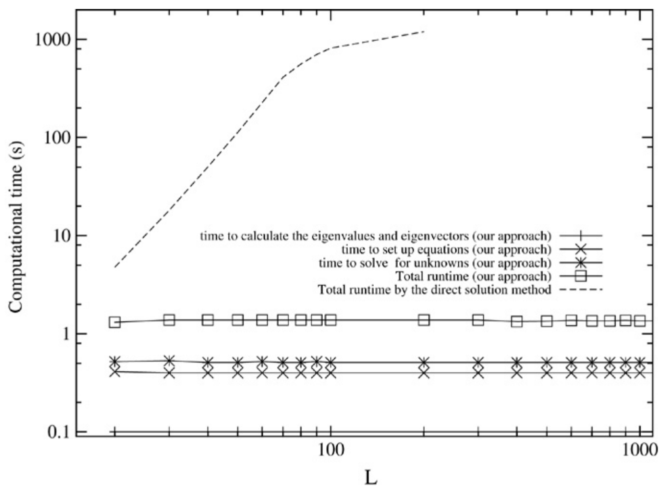

Fig. 1. Comparison of runtimes.

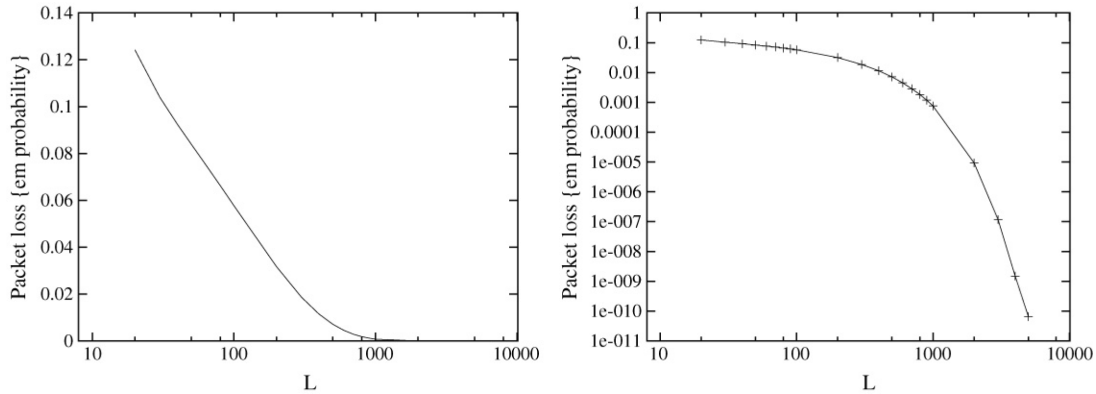

Fig. 2. Packet loss probabilities (y-axis on the linear and log scales respectively).

proposed method:

- The time needed to calculate the eigensystem for the spectral expansion method and the time needed to solve the resulting linear simultaneous equations for the unknowns by our method (see Section 5) are constant and do not depend on L.
- However, we have to set up the simultaneous equations to find the unknowns. The time needed to set up the simultaneous equations based on our method does not depend on L either.
- The runtime needed to compute the steady state probabilities and performance measures by the direct solution method is also illustrated in the figure. For example, the runtime required by our method is 1.38 s, practically independent of L. The runtimes by the direct solution are L-dependent, for example, 817 s for $L = 100$, and 1200 s for $L = 200$. For $L > 300$, the direct solution method was not able to produce results because there is not enough memory. When L is infinite, it is impossible to get results using the direct solution method, while our approach needs even less than 1.3 s (i.e., we need a similar amount of time as before to calculate the eigensystem and, in fact, less time to determine the unknowns since the number of unknowns is halved).

We also present the packet loss probability, in the considered scenario, versus L, in Fig. 2. Note that the curve can be used to determine the appropriate buffer size required in order to achieve the desired Quality of Service (QoS) level. As expected, the loss probability is asymptotic to zero. The relationship of large buffer and very small packet loss probability can also be observed (i.e., the curve of packet loss probability has different slopes).

7. Conclusions

One of the research aims in the performance evaluation of telecommunication networks is to find analytically and computationally tractable queueing models with the capability of capturing the burstiness and auto-correlations of the traffic. In this context, the first contribution of this paper is the introduction and the *exact and efficient* solution for the steady state probabilities of the MM $\sum_{k=1}^{K}$ CPP$_k$/GE/c/L G-queue with heterogeneous servers (the *HetSigma* queue). This queue is capable of capturing the burstiness and auto-correlations of the traffic, and accommodating large or unbounded batch-sizes while also taking into account environmentally sensitive service times. In fact, we generalise significantly the previous works related to the arrival process (i.e., MMPP, CPP, MMCPP) and the service time distribution (i.e., GE service times). Moreover, the inclusion of negative customers in the model provides a flexible platform or framework for studies related to modelling packet-losses, unreliable servers and load-balancing. Further research may involve (i) possible extensions to this model with alternate scheduling or killing disciplines to suit different applications, (ii) further evolution of the model towards much more *generalised Markovian node models* for Advanced Computing Systems (ACS) and Next Generation Networks (NGN).

Secondly, we also illustrate the application of this queue to the performance evaluation of a wireless communication system. Numerical study is presented to illustrate the efficacy and computational efficiency of the proposed method.

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Appendix A. Obtaining the Q_l matrices

Consider any row j where $c + K + 1 \le j \le L - 2 - c$. With Transformation 1, we get

$$\langle \mathbf{j} \rangle^{(1)} \longleftarrow \langle \mathbf{j} \rangle + \sum_{l=1}^{K} (-1)^l \langle \mathbf{j} - \mathbf{l} \rangle F_{K,l}. \tag{A.1}$$

Applying Transformation 2 to the jth row, from the above (A.1), we get

$$\langle \mathbf{j} \rangle^{(2)} \longleftarrow \langle \mathbf{j} \rangle^{(1)} + \sum_{n=1}^{c} (-1)^n \langle \mathbf{j} + \mathbf{n} \rangle^{(1)} H_{c,n}. \tag{A.2}$$

Expanding the terms, Eq. (A.2) can be written as

$$\begin{aligned} \langle \mathbf{j} \rangle^{(2)} \longleftarrow & \langle \mathbf{j} \rangle + \sum_{l=1}^{K} (-1)^l \langle \mathbf{j} - \mathbf{l} \rangle F_{K,l} \\ & + \sum_{n=1}^{c} (-1)^n \left[\langle \mathbf{j} + \mathbf{n} \rangle + \sum_{l=1}^{K} (-1)^l \langle \mathbf{j} - \mathbf{l} + \mathbf{n} \rangle F_{K,l} \right] H_{c,n}. \end{aligned} \tag{A.3}$$

Applying Transformation 3 to the jth row, and substituting from the above (A.3), for $\langle \mathbf{j} + \mathbf{1} \rangle^{(2)}$,

$$\begin{aligned} \langle \mathbf{j} \rangle^{(3)} \longleftarrow & \langle \mathbf{j} \rangle + \sum_{l=1}^{K} (-1)^l \langle \mathbf{j} - \mathbf{l} \rangle F_{K,l} \\ & + \sum_{n=1}^{c} (-1)^n \left[\langle \mathbf{j} + \mathbf{n} \rangle + \sum_{l=1}^{K} (-1)^l \langle \mathbf{j} - \mathbf{l} + \mathbf{n} \rangle F_{K,l} \right] H_{c,n} \\ & - \left[\langle \mathbf{j} + \mathbf{1} \rangle + \sum_{l=1}^{K} (-1)^l \langle \mathbf{j} + \mathbf{1} - \mathbf{l} \rangle F_{K,l} \right] \Delta \end{aligned}$$

$$-\sum_{n=1}^{c}(-1)^n\left[\langle\mathbf{j}+\mathbf{1}+\mathbf{n}\rangle+\sum_{l=1}^{K}(-1)^l\langle\mathbf{j}+\mathbf{1}-\mathbf{l}+\mathbf{n}\rangle F_{K,l}\right]H_{c,n}\Delta \tag{A.4}$$

Expanding and grouping the terms together, Eq. (A.4) can be written as

$$\langle\mathbf{j}\rangle^{(\mathbf{3})}\longleftarrow\sum_{m=-c-1}^{K}\langle\mathbf{j}-\mathbf{m}\rangle G_{K,c,m} \tag{A.5}$$

where

$$\begin{aligned}G_{K,c,m}&=\sum_{\substack{l-n=m\\ l=-1,\dots,K\\ n=0,\dots,c}}(-1)^{l+n}[F_{K,l}H_{c,n}+F_{K,l+1}H_{c,n}\Delta]\\ &=\sum_{n=0}^{c}(-1)^{m+2n}[F_{K,m+n}+F_{K,m+n+1}\Delta]H_{c,n}\\ &=(-1)^m\sum_{n=0}^{c}[F_{K,m+n}+F_{K,m+n+1}\Delta]H_{c,n}\quad(m=-1-c,\dots,K).\end{aligned} \tag{A.6}$$

The balance equations $\langle\mathbf{j}+\mathbf{c}+\mathbf{1}\rangle,\dots,\langle\mathbf{j}\rangle,\dots,\langle\mathbf{j}-\mathbf{l}\rangle,\dots,\langle\mathbf{j}-\mathbf{K}\rangle$, respectively are given by

$$\sum_{s=1}^{j+c+1}\sum_{k=1}^{K}\mathbf{v}_{j+c+1-s}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k+\mathbf{v}_{j+c+1}\left[Q-\Sigma-C_{j+c}-R\right]$$
$$+\sum_{s=1}^{L-j-c-1}\mathbf{v}_{j+c+1+s}C_{j+c+1+s,j+c+1}=0;$$

$\vdots$

$$\sum_{s=1}^{j}\sum_{k=1}^{K}\mathbf{v}_{j-s}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k+\mathbf{v}_j[Q-\Sigma-C_j-R]+\sum_{s=1}^{L-j}\mathbf{v}_{j+s}C_{j+s,j}=0;$$

$\vdots$

$$\sum_{s=1}^{j-l}\sum_{k=1}^{K}\mathbf{v}_{j-l-s}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k+\mathbf{v}_{j-l}\left[Q-\Sigma-C_{j-l}-R\right]+\sum_{s=1}^{L-j+l}\mathbf{v}_{j-l+s}C_{j-l+s,j-l}=0;$$

$\vdots$

$$\sum_{s=1}^{j-K}\sum_{k=1}^{K}\mathbf{v}_{j-K-s}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k+\mathbf{v}_{j-K}[Q-\Sigma-C_{j-K}-R]+\sum_{s=1}^{L-j+K}\mathbf{v}_{j-K+s}C_{j-K+s,j-K}=0;$$

Substituting or applying the above to (A.5), for the coefficients (Q_{K-m}) of $\mathbf{v}_{j-m}$ in $\langle\mathbf{j}\rangle^{(\mathbf{3})}$, we get

$$\begin{aligned}Q_{K-m}&=\sum_{l=-1-c}^{m-1}\left[\sum_{n=1}^{K}\Theta_n^{m-l-1}(E-\Theta_n)\Sigma_n\right]G_{K,c,l}+[Q-\Sigma-C_{j-m}-R]G_{K,c,m}\\ &\quad+\sum_{l=m+1}^{K}[C_{j-m,j-l}]G_{K,c,l}\quad(m=j-L,\dots,-2,-1,0,\dots,K,\dots,j).\end{aligned} \tag{A.7}$$

Also, for $m=-1-c,\dots,0,\dots,K$, substituting $C_{j-m}=C$ and $C_{j-m,j-l}=C_{j-l+l-m,j-l}=\sum_{n=1}^{c}M_n(E-\Phi_n)\Phi_n^{l-m-1}+R(E-\Delta)\Delta^{l-m-1}$ in (A.7), we get

$$Q_{K-m}=\sum_{l=-1-c}^{m-1}\left[\sum_{n=1}^{K}\Theta_n^{m-l-1}(E-\Theta_n)\Sigma_n\right]G_{K,c,l}+[Q-\Sigma-C-R]G_{K,c,m}$$

Table B.1
Mathematica code to symbolically verify that the Q_l ($l < 0$ or $l > K + c + 1$) are zero

```
(*This is a Q_{K - m}*) QPolyCoeff[NN_,K_,c_,m_]:=Block[{},
 Q=Array[q,{NN,NN}];
 For[i = 1, i <= NN,
    i++, {seged = Sum[q[i, j], {j, 1, NN}] - q[i, i]; q[i, i] = -seged }];
               (*this to compute the diagonal elements of the Q matrix*)
 TheSigmaMatrix=Array[sigmaM,{NN,K}];
 TheThetaMatrix=Array[thetaM,{NN,K}];
 For[i=1,i<=K,i++,Sigma[i]=DiagonalMatrix[Transpose[TheSigmaMatrix][[i]]]];
 SumSigma=Sum[Sigma[i],{i,1,K}];
 For[i=1,i<=K,i++,Theta[i]=DiagonalMatrix[Transpose[TheThetaMatrix][[i]]]];
 TheMuMatrix=Array[mu,{NN,c}];
 ThePhiMatrix=Array[phiM,{NN,c}];

 For[i=1,i<=c,i++,M[i]=DiagonalMatrix[Transpose[TheMuMatrix][[i]]]];
 Cmatrix=Sum[M[i],{i,1,c}];
 For[i=1,i<=c,i++,Phi[i]=DiagonalMatrix[Transpose[ThePhiMatrix][[i]]]];
 Delta=DiagonalMatrix[Array[deltaM,NN]];
 R=DiagonalMatrix[Array[rho,NN]];
 H[1,-1]=IdentityMatrix[NN]-IdentityMatrix[NN];
 F[k_,l_]:=If[l>=0&&l<=k,F[k-1,l]+Theta[k].F[k-1,l-1],H[1,-1]];
 F[1,0]=IdentityMatrix[NN];
 F[1,1]=Theta[1];F[1,-1]=IdentityMatrix[NN]-IdentityMatrix[NN];
 H[k_,l_]:=If[l>=0&&l<=k,H[k-1,l]+Phi[k].H[k-1,l-1],H[1,-1]];
 H[1,0]=IdentityMatrix[NN];
 H[1,1]=Phi[1];
 H[1,-1]=IdentityMatrix[NN]-IdentityMatrix[NN];
 G[k_]:=Power[-1,k] Sum[(F[K,k+n]+F[K,k+n+1] Delta) H[c,n],{n,0,c}];
 Return[Sum[MatrixPower[Theta[n],m-l-1].
          (IdentityMatrix[NN]-Theta[n]).Sigma[n]. G[l],{l,-1-c,m-1},{n,1,K}]+
          (Q-SumSigma-Cmatrix-R).G[m]+
        Sum[(Sum[M[n].(IdentityMatrix[NN]-Phi[n]).
         MatrixPower[Phi[n],l-m-1],{n,1,c}]+
        R.(IdentityMatrix[NN]-Delta).MatrixPower[Delta,l-m-1]).G[l],{l,m+1,K}]]
];

NN = 3; K = 3; c = 4;
Print[Simplify[QPolyCoeff[NN, K, c, K +1]]];
{{0, 0, 0},{0,0,0},{0, 0, 0}}
Print[Simplify[QPolyCoeff[NN, K,c,K+2]]];
{{0,0,0},{0,0,0},{0,0,0}}
Print[Simplify[QPolyCoeff[NN,K,c,-c-2]]];
{{0,0,0},{0,0,0},{0,0,0}}
Print[Simplify[QPolyCoeff[NN, K, c, -c -3]]];
{{0,0,0},{0,0,0},{0,0,0}}


NN = 2; K = 2; c = 2;
Print[Simplify[QPolyCoeff[NN, K, c, K +1]]];
{{0,0},{0,0}}
Print[Simplify[QPolyCoeff[NN, K, c, K + 2]]];
{{0,0},{0,0}}
Print[Simplify[QPolyCoeff[NN, K, c, -c - 2]]];
{{0,0},{0,0}}
Print[Simplify[QPolyCoeff[NN, K, c, -c - 3]]];

{{0,0},{0,0}}
```

This program works for *any values of* N, K *and* c (NN is used because N is a reserved keyword in *Mathematica*).

$$
+ \sum_{l=m+1}^{K} \left[\sum_{n=1}^{c} M_n (E - \Phi_n) \Phi_n^{l-m-1} + R(E - \Delta)\Delta^{l-m-1} \right] G_{K,c,l} \quad (m = -1 - c, \ldots, 0, \ldots, K). \tag{A.8}
$$

Using the above, the required Q_l's can be computed easily as described in the subsection below. Notice that the above Q_l's in Eq. (A.8) are j-independent.

The other coefficients, i.e., those of $\mathbf{v}_{j-K-1}, \mathbf{v}_{j-K-2}, \ldots, \mathbf{v}_0$ and of $\mathbf{v}_{j+c+2}, \mathbf{v}_{j+c+3}, \ldots$, can be shown to be zero.

Table B.2
Mathematica code to automatically and symbolically verify that $\sum_{m=-1-c}^{k} Q_{k-m}^{(k)}$ is singular

```
SS[k_, h_] := Sum[QPolyCoeff[NN, k, h, m], {m, -h - 1, k}];

NN = 2; K = 2; c = 2;
Simplify[Det[SS[K, c]]];
Out[99]=0

NN = 4; K = 4; c = 3;
Simplify[Det[SS[K, c]]];
Out[100]=0
```

Note that is valid for any values of N, K and c.

Computation

After obtaining $F_{K,l}$'s and $H_{c,n}$'s thus, $G_{K,c,m}$, $(m = -1-c, \ldots, K)$ can be computed from (A.6). Then, using them directly in (A.8), the required Q_l $(l = 0, 1, \ldots, K + c + 1)$ can be computed.

Appendix B. Automatic validation of equations

In this section we present some theorems with empirical validation, using *Mathematica*. Rigorous proof can be done as in [16], but they are not presented here because of page limitation. The theorems can be symbolically validated for any values of N, K and c by *Mathematica*.

Since, K, c are themselves arbitrary in this section, let the Q_l's be designated differently to take that into account. Let $Q_{k-m}^{(k,h)}$ $(m = j-L, j-L+1, \ldots, j)$ be the Q_l's of $\langle \mathbf{j} \rangle^{(3)}$ when only the first k customer arrival streams and the first h servers are present and others are absent.

Theorem 1. *Referring to Eq. (A.7) for the row* j $(c + K + 1 \le j \le L - 2 - c)$, *for all* K, $Q_{K-m}^{(K,c)} = 0$ $(j - L \le m \le -2 - c)$.

Theorem 2. *Referring to Eq. (A.7) for the row* j $(K + 1 \le j \le L - 2 - c)$, *for all* K, $Q_{K-m}^{(K,c)} = 0$ $(K + 1 \le m \le j)$.

The *Mathematica* code for the validation of Theorems 1 and 2 is illustrated in Table B.1.

Theorem 3. *Referring to Eq. (A.7) for the row* j $(K+1 \le j \le L-2-c)$, *for all* K, $Q_{K-m}^{(K,c)}$ $(m = -1-c, 0, \ldots, K)$ *are* j*-independent.*

Proof. $Q_{K-m}^{(K,c)}$ for $m = -1 - c, 0, \ldots, K$ are separately derived in (A.8). From the R.H.S. of (A.8), it is clear that $Q_{K-m}^{(K)}$ $(m = -1 - c, \ldots, K)$ are j-independent

Theorem 4. *Referring to Eq. (A.7) for the row* j $(K + 1 \le j \le L - 2 - c)$, *for all* K, c, $S_{K,c} = \sum_{m=-1-c}^{K} Q_{K-m}^{(K,c)}$ *is singular.*

The *Mathematica* code for the validation of Theorem 4 is presented in Table B.2. It can be seen that the determinant of $\sum_{m=-1-c}^{K} Q_{K-m}^{(K,c)}$ is zero.

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