An Efficient Computation Algorithm for A Multiserver Feedback Retrial Queue With A Large Queueing Capacity

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Abstract

Kumar et al. consider the M/M/c/N+c feedback queue with constant retrial rate [1]. They provide a solution for the steady state probability based on the matrix-geometric method. We show that there exists a more efficient computation method to calculate the steady state probabilities when $N + c$ is large. We prove that the number of zero-eigenvalues of the characteristic matrix polynomial associated with the balance equation is $\lfloor (N + c + 2)/2 \rfloor$. As consequence, the remaining eigenvalues inside the unit circle can be computed in a quick manner based on the Sturm sequences. Therefore, the steady state probabilities can be determined in an efficient way.

1 Introduction

The concept of Quasi Birth-Death (QBD) processes, as a generalization of the classical birth and death M/M/1 queues was first introduced by [2] and [3] in the late sixties. The states of a QBD process are described by two dimensional random variables called a phase and a level [4–6] and transitions in a QBD process are only possible between adjacent levels. It is observed that QBD processes create a useful framework for the performability analysis of many problems in telecommunications and computer networks [1,7–15].

There are two main methods to find the steady state probabilities for QBD processes on semi-infinite strips. The matrix-geometric [4] method (which is widely

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used to analyze queues [7,16]) and its variants [6,12] are numerical approaches to recursively compute the rate matrix (the minimal nonnegative matrix solution) of the matrix quadratic equation. The spectral expansion method is based on the eigenvalues and eigenvectors of the characteristic matrix polynomial [5,17]. It is confirmed by a number of works that the spectral expansion method is better than the matrix geometric one from some aspects [5,18,19]. It is worth emphasizing that there have been no works which consider or develop an algorithm for queues involving zero-eigenvalues (see [5,8,20–22] for more detail). In this paper, we deal with an algorithm to find the eigenvalues of the matrix quadratic equation of a tridiagonal form based on the sign variations of the Sturm sequences. We consider a case where multiple zero-eigenvalues are involved.

The example for investigation is the M/M/c/N+c feedback queue with constant retrial rate, which is solved by the matrix-geometric method in [1]. However, the existing approaches face the state explosion problem when the queueing capacity ($N + c$) is large. We prove that the number of zero-eigenvalues of the characteristic matrix polynomial associated with the balance equation is $\lfloor (N + c + 2)/2 \rfloor$. As consequence, the remaining eigenvalues inside the unit circle can be computed in a quick manner based on the Sturm sequences. It is worth emphasizing that the algorithm of [22] should be slightly modified in order to determine the remaining eigenvalues of the characteristic matrix polynomial inside the unit circle. Numerical results are presented to compare computation times which are needed by the matrix geometric method, the pure spectral expansion approach, the method proposed by Naoumov et al. and the new algorithm to calculate the steady state probabilities. The comparison clearly demonstrates the advantage of the new algorithm on the computation of the steady state probabilities of the queue with a large queueing capacity.

The rest of this paper is organized as follows. In Section 2, the M/M/c/N+c feedback queue with constant retrial rate is described. A main theoretical result concerning the number of zero-eigenvalues of the characteristic matrix polynomial associated with the balance equation and the modified algorithm to calculate the eigenvalues is presented in Section 3. Numerical results are demonstrated in Section 4. Finally, Section 5 concludes the paper.

2 A system description

The M/M/c/N+c feedback queue with constant retrial rate has a limited waiting position of size N and c homogeneous servers. Service times are exponentially distributed with parameter μ. External customers arrive according to a Poisson process with rate λ. Upon the arrival, an external customer

- either is served if there is a free server,

- or occupies a waiting position if all c servers are busy and there is a free waiting position,
- or is blocked and is forced to leave the system forever if all c servers are busy and waiting positions are occupied.

When one of the servers becomes free, the customer in the first waiting position immediately starts getting served.

A customer who leaves a system after service either join the retrial group (orbit) for another service with probability q ($0 \leq q < 1$) or leave the system forever with probability $p = 1 - q$. Customers in the retrial group request service with constant retrial rate σ, which is independent of the number of customers in the retrial group [1]. Note that a customer in the orbit can enter the service facility at the retrial instant only when there are idle servers.

The system is ergodic if the following expression holds (c.f. [1])

$$q\mu \left[\sum_{l=1}^{c-1} \frac{\left(\frac{\lambda+\sigma}{\mu}\right)^l}{(l-1)!} + \frac{\left(\frac{\lambda+\sigma}{\mu}\right)^c}{(c-1)!} \left(\frac{1 - (\frac{\lambda}{c\mu})^{N+1}}{1 - \frac{\lambda}{c\mu}} \right) \right] < \sigma \sum_{l=1}^{c-1} \frac{(\frac{\lambda+\sigma}{\mu})^l}{l!}. \quad (1)$$

The system is completely described by two random variables. $I(t)$ ($0 \leq I(t) \leq c + N$) is a random variable to denote the number of customers in the system (being served by servers or waiting in the waiting positions). $J(t)$ ($J(t) \geq 0$) is a random variable to represent the number of customers in the retrial group. $Y = \{I(t), J(t)\}$ is a Continuous Time Markov Chain (CTMC) with a state space $\{0, 1, \ldots, c + N\} \times \{0, 1, \ldots\}$. We denote the steady state probabilities by $\pi_{i,j} = \lim_{t\to\infty} Prob(I(t) = i, J(t) = j)$, and introduce $\mathbf{v}_j = (\pi_{0,j}, \ldots, \pi_{c+N,j})$.

The evolution of Y is driven by the following transitions.

(a) $A_j(i, k)$ denotes a transition rate from state (i, j) to state (k, j) ($0 \leq i, k \leq n = c + N; j = 0, 1, \ldots$). These transitions happen due to either the arrival of an external customer or the departure of a customer from the system. $A_j(i, k)$ does not depend on j, so we can write

$$A_j(i, k) = A(i, k) = \begin{cases} p\mu \min(i, c) & \text{if } k = i - 1 \text{ and } i \geq 1 \\ \lambda & \text{if } k = i + 1 \\ 0 & \text{otherwise} \end{cases} . \quad (2)$$

(b) $B_j(i, k)$ represents one step upward transition from state (i, j) to state $(k, j + 1)$ ($0 \leq i, k \leq n = c+N; j = 0, 1, \ldots$). These transitions are due to customers who join the retrial group.

$B_j(i,k)$ is independent of j, thus it is valid

$$B_j(i,k) = B(i,k) = \begin{cases} q\mu \min(i,c) & \text{if } k = i-1 \text{ and } i \geq 1 \\ 0 & \text{otherwise} \end{cases} . \quad (3)$$

(c) $C_j(i,k)$ is the transition rate from state (i,j) to state $(k,j-1)$ $(0 \leq i,k \leq n = c+N; j = 1,\ldots)$. These transitions are initiated by requests from the retrial group. $C_j(i,k)$ does not depend on j, so we can write

$$C_j(i,k) = C(i,k) = \begin{cases} \sigma & \text{if } k = i+1 \text{ and } i < c \\ 0 & \text{otherwise} \end{cases} . \quad (4)$$

$$A = \begin{bmatrix} 0 & \lambda & 0 & 0 & 0 & 0 \\ p\mu & 0 & \lambda & 0 & 0 & 0 \\ 0 & 2p\mu & 0 & \lambda & 0 & 0 \\ 0 & 0 & 2p\mu & 0 & \lambda & 0 \\ 0 & 0 & 0 & 2p\mu & 0 & \lambda \\ 0 & 0 & 0 & 0 & 2p\mu & 0 \end{bmatrix}; Q_0 = B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ q\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 2q\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 2q\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2q\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2q\mu & 0 \end{bmatrix};$$

$$Q_2 = C = \begin{bmatrix} 0 & \sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$Q_1 = \begin{bmatrix} -\lambda-\sigma & \lambda & 0 & 0 & 0 & 0 \\ p\mu & -\lambda-\mu-\sigma & \lambda & 0 & 0 & 0 \\ 0 & 2p\mu & -\lambda-2\mu & \lambda & 0 & 0 \\ 0 & 0 & 2p\mu & -\lambda-2\mu & \lambda & 0 \\ 0 & 0 & 0 & 2p\mu & -\lambda-2\mu & \lambda \\ 0 & 0 & 0 & 0 & 2p\mu & -2\mu \end{bmatrix};$$

$$Q(x) = \begin{bmatrix} (-\lambda-\sigma)x & \lambda x+\sigma x^2 & 0 & 0 & 0 & 0 \\ q\mu+p\mu x & (-\lambda-\mu-\sigma)x & \lambda x+\sigma x^2 & 0 & 0 & 0 \\ 0 & 2q\mu+2p\mu x & (-\lambda-2\mu)x & \lambda x & 0 & 0 \\ 0 & 0 & 2q\mu+2p\mu x & (-\lambda-2\mu)x & \lambda x & 0 \\ 0 & 0 & 0 & 2q\mu+2p\mu x & (-\lambda-2\mu)x & \lambda x \\ 0 & 0 & 0 & 0 & 2q\mu+2p\mu x & -2\mu x \end{bmatrix}.$$

Table 1
Matrices for $c = 2$ and $N = 3$

$A(i,k)$, $B(i,k)$ and $C(i,k)$ are the elements of A, B and C matrices, respectively. We introduce diagonal matrices D^A, D^B and D^C. The diagonal elements are the sum of the elements in the row of A, B and C. The infinitesimal generator matrix of Y can be written as follows:

$$\begin{bmatrix} A_{00} & Q_0 & 0 & \dots & \dots & \dots & \dots \\ Q_2 & Q_1 & Q_0 & 0 & \dots & \dots & \dots \\ 0 & Q_2 & Q_1 & Q_0 & 0 & \dots & \dots \\ 0 & 0 & Q_2 & Q_1 & Q_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}, \tag{5}$$

where $A_{00} = A - D^A - D^B$, $Q_0 = B$, $Q_1 = A - D^A - D^B - D^C$ and $Q_2 = C$. The forms of the matrices are illustrated in Table 1.

3 An efficient computation algorithm

3.1 The number of zero-eigenvalues

For $j \geq 1$, the balance equations is written as follows:

$$\mathbf{v}_{j-1}Q_0 + \mathbf{v}_j Q_1 + \mathbf{v}_{j+1}Q_2 = 0 \ \ (j \geq 1). \tag{6}$$

$Q(x) = Q_0 + Q_1 x + Q_2 x^2$ is defined as the characteristic matrix polynomial associated with equations (6). In the present paper, $Q(x)$ is a tridiagonal matrix

$$Q(x) = \begin{bmatrix} q_{0,0}(x) & q_{0,1}(x) & 0 & \dots & 0 & 0 & 0 \\ q_{1,0}(x) & q_{1,1}(x) & q_{1,2}(x) & \dots & 0 & 0 & 0 \\ 0 & q_{2,1}(x) & q_{2,2}(x) & q_{2,2}(x) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & \dots & q_{n-1,n-2}(x) & q_{n-1,n-1}(x) & q_{n-1,n}(x) \\ 0 & 0 & & \dots & 0 & q_{n,n-1}(x) & q_{n,n}(x) \end{bmatrix} \tag{7}$$

where

$$
\begin{aligned}
q_{0,0}(x) &= -(\lambda+\sigma)x, \\
q_{i,i-1}(x) &= q\mu\min(i,c) + p\mu\min(i,c)x \qquad (i=1,\ldots,n), \\
q_{i,i}(x) &= -(\lambda + \mu\min(i,c) + C(i,i+1))x \qquad (i=1,\ldots,n-1), \\
q_{i,i+1}(x) &= \lambda x + C(i,i+1)x^2 \qquad (i=0,\ldots,n-1), \\
q_{n,n}(x) &= -c\mu x.
\end{aligned}
$$

The steady state probabilities are closely related to the eigenvalue-eigenvector pairs $(x_i, \boldsymbol{\psi}_i)$ of $Q(x)$, which satisfy $\boldsymbol{\psi}Q(x) = 0$ and $det[Q(x)] = 0$ (c.f. [5,23]). If the system is ergodic (which is so when inequation (1) holds), then the number of eigenvalues of the characteristic polynomial $Q(x)$ with *a degree of* $n+1$, which are strictly inside the unit disk, has to be $n+1$ (see the proof in [23]). So we can write

$$
\mathbf{v}_j = \sum_{i=0}^{n} a_i x_i^j \boldsymbol{\psi}_i \quad (j \geq 0), \tag{8}
$$

where x_i are the eigenvalues inside the unit circle. Coefficients a_i can be determined from the balance equation for level $j=0$ and the normalization equation.

Theorem 1 *The number of zero-eigenvalues of* $Q(x)$ *is* $\lfloor (n+2)/2 \rfloor$.

Proof. We provide a proof with mathematical induction. Let $[Q(x)]_{\{0,\ldots,k\}}$ denote a submatrix formed by the first $k+1$ rows and columns of $Q(x)$. It is easy to verify that $det[Q(x)]_{\{0,1\}} = 0$ has one zero-root and $det[Q(x)]_{\{0,1,2\}} = 0$ has two zero-roots.

Assume that $det[Q(x)]_{\{0,\ldots,k-2\}} = 0$ ($k \geq 3$) has $\lfloor k/2 \rfloor$ zero-roots and $det[Q(x)]_{\{0,\ldots,k-1\}} = 0$ has $\lfloor (k+1)/2 \rfloor$ zero-roots.

$[Q(x)]_{\{0,\ldots,k\}}$ is a tridiagonal matrix. Therefore, we can write

$$
det[Q(x)]_{\{0,\ldots,k\}} = q_{k,k}(x)det[Q(x)]_{\{0,\ldots,k-1\}} - q_{k,k-1}(x)q_{k-1,k}(x)det[Q(x)]_{\{0,\ldots,k-2\}}.
$$

Note that $q_{k,k}(x)$ and $q_{k-1,k}(x)$ is divideable by x, while $q_{k,k-1}(x)$ is not. As a consequence, $det[Q(x)]_{\{0,\ldots,k\}} = 0$ has $\lfloor k/2 \rfloor + 1 = \lfloor (k+2)/2 \rfloor$ zero-roots. □

3.2 A computation algorithm

Following [22], $\boldsymbol{\psi}Q(x) = 0$ can be written as,

$$
\begin{aligned}
0 &= \psi_0 q_{0,0}(x) + \psi_1 q_{1,0}(x), \\
0 &= \psi_{i-1} q_{i-1,i}(x) + \psi_i q_{i,i}(x) + \psi_{i+1} q_{i+1,i}(x), \quad i = 1, \dots, n-1 \\
0 &= \psi_{n-1} q_{n-1,n}(x) + \psi_n q_{n,n}(x),
\end{aligned}
$$

where $\boldsymbol{\psi} = \{\psi_0, \dots, \psi_n\}$.

We set $\psi_0 = 1$ and $q_{n+1,n} = 1 + x$, therefore

$$
\begin{aligned}
\psi_1(x) &= -q_{0,0}(x)/q_{1,0}(x), \\
\psi_{i+1}(x) &= -\frac{\psi_i(x) q_{i,i}(x) + \psi_{i-1}(x) q_{i-1,i}(x)}{q_{i+1,i}(x)}, i = 1, \dots, n
\end{aligned} \tag{9}
$$

It is proved by [22] that the sequence $\{\psi_i(x), i = 0, \dots, n+1\}$ associated with the characteristic matrix polynomial of tridiagonal form is a Sturm sequence within a given interval if for any fixed x within this interval $\psi_0(x) = 1$ and $\psi_i(x) = 0$, $i = 1, \dots, n$ implies $\psi_{i-1}(x)\psi_{i+1}(x) < 0$. Furthermore, the number of sign variations is defined by [22]

$$
nsv(x) = \#\{\psi_i(x)\psi_{i+1}(x) < 0, 0 \leq i \leq n\} + \#\{\psi_i(x) = 0, 0 \leq i \leq n\}. \tag{10}
$$

Grassmann [22] has reported a divide-and-conquer procedure (called `getx` in Algorithm 1) to find eigenvalues inside the unit circle for QBD processes with the characteristic polynomial matrix of a tridiagonal form if they are all non-zero. Grassmann's algorithm discards any interval $(x_1, x_2]$ if $nsv(x_1) = nsv(x_2)$. To compute the eigenvalues in interval $(0, 1)$, it is proposed to start the algorithm with `getx(0,n+1,1,0)` (c.f.: [22]). It is also mentioned [22] that handling zero-eigenvalues will be developed in future.

If the QBD process of the M/M/c/N+c feedback queue with constant retrial rate is ergodic, then the number of eigenvalues inside the unit circle is $n + 1$. We have proved that the number of zero-eigenvalues is $\lfloor (n+2)/2 \rfloor$. In order to deal with zero-eigenvalues, two modifications are needed.

- the modified function for the number of sign variations is defined as follows:

$$
mnsv(x) = \#\{\psi_i(x)\psi_{i+1}(x) < 0, 0 \leq i \leq n\}. \tag{11}
$$

  The new function for the number of sign variations should be applied inside the `getx` function as well.
- the modified initialization can be applied as illustrated in Algorithm 2 to find the remaining non-zero eigenvalues of $Q(x)$. Note that $mnsv(\epsilon) = n - \lfloor n/2 \rfloor$ because the number of other eigenvalues inside the unit circle is $n - \lfloor n/2 \rfloor$.

Algorithm 1 `getx` procedure

{Xeg is the vector of eigenvalues} {ϵ is the required accuracy}

PROCEDURE getx(x_1, nx_1, x_2, nx_2)
if $nx_1 == nx_2$ **then**
  Return
end if
if $x_2 - x_1 < \epsilon$ **then**
  if $nx_1 == nx_2 + 1$ **then**
    $Xeg_{nx_2} = x_1$
  end if
  Return
end if
$x = \frac{x_1+x_2}{2}$
$nx = nsv(x)$
Call getx(x_1, nx_1, x, nx)
Call getx(x, nx, x_2, nx_2)
END OF PROCEDURE getx

Note that for each eigenvalue, the corresponding eigenvector can be determined with equation (9), then the steady state probabilities can be computed.

4 Numerical results

In this Section, we illustrate the efficiency of the computation method for large queueing capacity vs other methods (direct computation of the eigenvalues of the characteristic matrix polynomial and the matrix geometric method). As already analyzed in [17], both the spectral expansion and the matrix-geometric method have the same complexity of solving the unknowns after the computation of eigenvalue-eigenvectors and the rate matrix. Therefore, we compare the computational time needed to obtain the eigenvalues/eigenvectors (for the new method and the direct calculation of the eigenvalues) and the rate matrix in the case of the matrix-geometric method. Four methods are compared in this section: the original spectral expansion method [23], the successive substitution procedure of the matrix-geometric method [4] proposed by Kumar et al. [1] for the M/M/c/N+c feedback queue with constant retrial rate, the variant of the matrix-geometric method [12] (which is considered as one of the best and latest improvements for the original matrix-geometric method) and the new algorithm.

For the numerical study, we choose the following parameters $N = 3$, $\sigma = 2.0$, $\mu = 1.0/c$, $p = 0.6$, $q = 0.4$. The accuracy is $\epsilon = 10^{-10}$. This value is also the stopping criteria for the matrix geometric approach. Results were produced in a machine with Intel® Xeon® E5410 2.33GHz processor. In Figure 1, we plot the computation time versus the queueing capacity n and λ. In the plots the GEO, SPE, NAO and NEW denotes results obtained by successive substitution procedure, original spectral expansion, an improvement for the matrix-geometric method by Naoumov et al. and the new method, respectively. It is observed that the matrix geometric method (the successive substitution procedure and a procedure proposed by Naoumov et al.) takes the smallest time to compute the steady state probabilities when n is smaller than 300. Note that the difference between Naoumov's method and the successive substitution (SS) procedure is that the SS procedure involves a less computation effort (matrix multiplication, substraction and addition) in one iteration than Naoumov's method. However, the convergence of the SS procedure is slower than the Naoumov's method. When n is small, the computation effort is a dominant factor in the case of Naoumov's method, so we can observe that the computation time is higher than the SS procedure in the plots.

For $n < 50$, the original spectral expansion method needs less time than the new method. The new method outperforms other methods for $n > 300$, where the computation time of the successive substitution procedure jumps to very high. The difference is more than one order of magnitude. It is worth mentioning that the memory requirement of the new method is minimal (e.g., no need to store matrices).

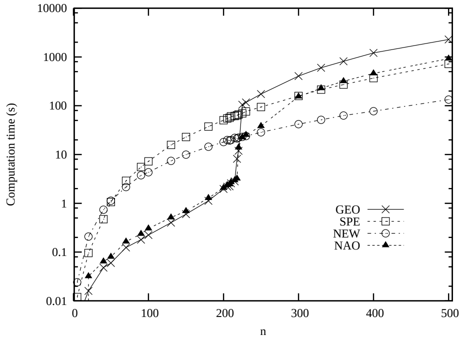
Algorithm 2 Initialization to find $n - \lfloor n/2 \rfloor$ non-zero eigenvalues inside the unit circle

{ ϵ is the required accuracy}
{Xeg is the vector of eigenvalues indexed from 0 to $n - \lfloor n/2 \rfloor$}
$x_1 := \epsilon$
$x_2 := 1 - \epsilon$
$nx_1 := mnsv(x_1)$
$nx_2 := mnsv(x_2)$
Call getx(x_1, nx_1, x_2, nx_2)

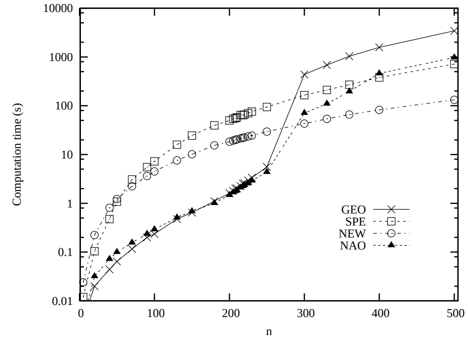
5 Conclusions

We have proved that the number of zero-eigenvalues of the characteristic matrix polynomial associated with the balance equation is $\lfloor (N + c + 2)/2 \rfloor$. We present an approach to deal with the multiple zero-eigenvalues. As consequence, the steady state probabilities are determined in an efficient way for the M/M/c/N+c feedback queue with a large queueing capacity.

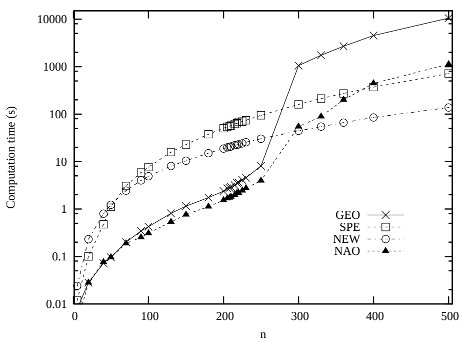
Numerical results are presented to compare computation times which are needed by the matrix geometric method, the pure spectral expansion approach, Naoumov's method and the new algorithm to calculate the steady state probabilities. The comparison clearly demonstrates the advantage of the new algorithm on the computation of the steady probability of the queue with large queueing capacity.



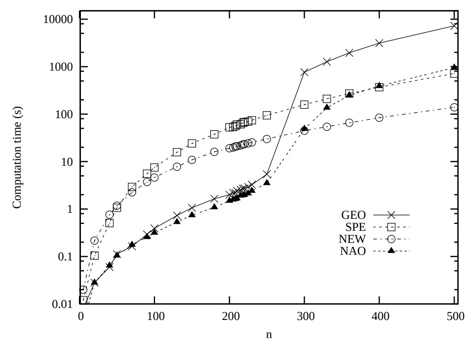
(a) $\lambda = 0.1$



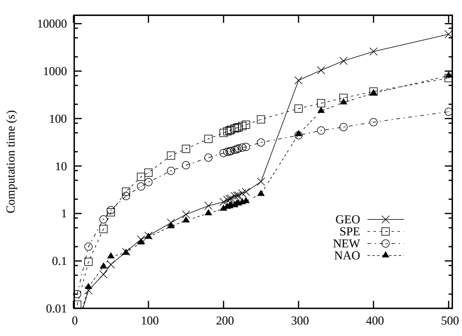
(b) $\lambda = 0.2$



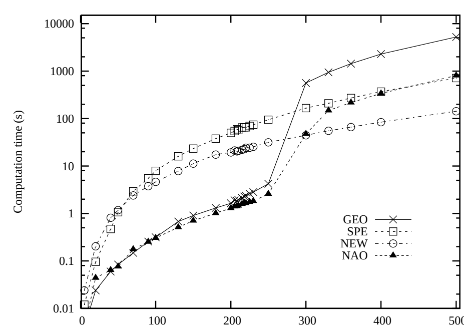
(c) $\lambda = 0.5$



(d) $\lambda = 0.9$



(e) $\lambda = 0.95$



(f) $\lambda = 0.99$

Fig. 1. Computation time in seconds

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