Non-Gaussian Option Pricing

Technical Report by Norbert Fogarasi (Jun 2003)

1. Introduction

There is now clear consensus amongst both academic researchers and industry practitioners that the Black-Scholes (BS) model in its simplest form (constant volatility) is a useful benchmark, but needs further enhancements to be useful in practice. In particular, Rubinstein (1994) observes that "there has been a very marked and rapid deterioration" since 1986 in the applicability of BS to S&P500 index options. He introduces the concept of the "volatility smile", i.e. the notion that the market prices of out-of-money options have a higher volatility implied by the BS model than in-the-money options. This gives rise to a downward sloping, convex pattern of BS implied volatilities for call options, when plotted against their exercise prices. Bates (1995) also provides a good empirical study of the BS anomalies.

In this chapter, we enumerate three distinct directions in option pricing research, which go beyond the strong assumptions of the BS model. The first direction is extending to the BS method by maintaining the assumption that the underlying process is governed by the Ito process:

$$\frac{dS}{S} = rdt + \sigma(t,S)dZ$$

but allow volatility to be non-constant. These models are characterized by the family of functions used for volatility, and include stochastic volatility, GARCH and jump diffusion models. Collectively, we refer to these as "Extensions of the BS Model". The second direction of research focuses on finding more realistic parameterizations of the underlying process, often substantiated by empirical analysis, and deriving the option pricing formulas implied by them. Indeed, one way to perceive the shortcoming of the BS model is to observe that the probabilty distribution of returns exhibited by various financial processes rarely display the Gaussian property, they are more realistically modeled by Levy alpha-stable processes, truncated Levy distributions, or distributions implied by the Langevin equation. The third, and most recently developed direction involves a nonparametric approach to the problem. These methods make no assumption about the underlying process, but apply various curve-fitting, learning network or regression techniques to obtain the option price as a function of underlying variables. Most recent approaches in this area incorporate various limitations implied by financial theory into the construction of the fitting technique, such as homogeneity, monotonicity or convexity.

2. Extensions of the BS Model

A natural extension to the BS model is to consider volatility to be a general function of the underlying price and time-to-maturity. A very simple extension introduced by Fogarasi et al (1998) in the interest rate derivatives framework is to consider volatility to be a piecewise linear deterministic function of time-to-maturity. It is shown that the resulting model can be calibrated to better fit market data, yet preserve the nice analytic properties of the BS model.

A simple extension in another direction is to consider volatility functions of the form $\sigma(S) = \sigma S^{a}$, which is known as the constant elasticity of variance (CEV) model. With a=0, this reduces to the BS case, a<0 implies that volatility increases as the underlying price decreases, while a>0 is vice versa. The CEV model also yields a closed form pricing formula for Eurpean call and put options.

Rubinstein (1994) observed that there is a negative correlation between the level of underlying and the implied volatility exhibited by option prices. This finding is important because it not only indicates that volatility must be considered a function of the underlying price, but also that it should be treated as stochastic. This gives rise to models where volatility is modelled as a continuous, stochastic process, separate from the evolution of the underlying, but possibly correlated. Examples of this “bivariate diffusion framework” are abundant: Hull and White (1987) Johnson and Shanno(1987), Scott (1987) are just a few examples. Substantial empirical work has also been performed on these models, eg. Bates (1996), Nandi (1998), which show them to have practical significance.

Another strand of literature develops option pricing models in discrete time, in a GARCH setting. Amin and Ng (1993), Duan (1995), Engle and Mustafa (1992) are all works in this area. It has been shown independently by Nelson (1990), Follmer and Schweizer (1991) and Duan (1996) that it is possible to think of GARCH and bivariate diffusion views of stochastic volatility in the same context: the discrete formulation converges to the continuous one under certain conditions.

In the implied volatility function (IVF) model, the volatility function is chosen in such a way that the model fits the prices observed today for all Europen option prices, regardless of the shape of the volatility surface. Both Dupire (1994) and Andersson and Brotherton-Ratcliffe (1997) derive an analytic formula for the volatility, as a function of strike price, time-to-maturity and Europen call prices observed in the market. This can then be used to construct a numerical option pricing model either using an implicit finite difference method or a so-called implied tree methodology, as in Derman and Kani (1994) or Rubinstein (1994).

3. Modeling the Underlying Process

The study of the dynamics of financial processes has a long history. The stochastic nature of the dynamics was observed as early as 1900 by Bachelier, who proposed the use of Brownian motion to model the stochastic process of returns over a small time scale. This approach is natural, if one considers the return over a small time scale to be the result of many independent shocks, which imply a Gaussian distribution of returns, by the central limit theorem.

The next important contribution to the topic was by Mandelbrot (1963), who not only showed that the probability distribution of returns has thicker tails than those implied by Gaussian distribution, but also that it has the self-similarity property. He observed that the statistical dependencies of returns have similar form for various time increments, ranging from 1 day to 1 month, much like fractals. Mandelbrot thus proposed that the distribution of returns is consistent with a Lévy stable distribution.

Most of the current research in this area build on Mandelbrot’s approach and model financial processes as fractals: non-differentiable, self-similar stochastic processes. Truncated Levy distributions are shown to exhibit most similar features to those observed empirically, and forms the subject of Mantega (1994), Koponen (1995), Bouchaud and Potters (2000). Mathematicians have developed a stochastic calculus for such non-Gaussian processes, as in Kleinert (2001), which has allowed option pricing theory to evolve in this framework as in Kleinert (2002), Matacz (2000) and Janecsko (2000).

Another recent development of note is the application of the Langevin equation, previously used in modeling fluid turbulence, which has also been shown to fit high-frequency financial data very well by Laskin (2000), Tang (2003). This work has not yet been extended to option pricing models.

4. Nonparametric Approach

A different approach from those described in section 3 is to disregard the study of the underlying process, and use a “model-free” approach. These methods are very much data-driven, and make very little assumptions about the underlying process or the derivative pricing model. They have the advantage of being robust to model-specification errors and are highly adaptive to the problem (and data) at hand. They are widely applicable to a range of derivative securities, yet simple to implement. On the other hand, they are highly data-intensive, and are highly dependent on the construction/architecture of the fitting technique used.

Several methods of function estimation techniques exist, including kernel-based methods, smoothing splines, orthogonal series estimators (such as Fourier series, Hermite polyinomials), genetic algorithms and neural networks.Ghysels et al (1996) provide a detailed summary of these methods.

Ait-Sahalia (1996) uses kernel-based methods to price interest rate derivative securities, while Ait-Sahalia and Duarte (2003) extend these methods while imposing shape restrictions implied by financial theory. They also show that these theoretical restrictions are actually violated in earlier studies which use unrestricted nonparametric pricing.

Stutzer (1996) and Duan (2002) take a different approach and attempt to develop kernel-based methods, which do not require any option data, only data for the underlying process, which extends their practical applicability. Thorough empirical work is yet to be done to validate these methods.

Genetic algorithms, inspired by biology use various combinations of operators (eg. moving averages, basic arithmetic operators etc) and crossing them over (like genes), eliminating the worst performing rules and mutating them again, up to a convergence point.

The application of Artificial Neural Networks as a nonparametric option pricing tool is one of the most fertile areas of current research. Developments in this area are discussed in a separate report.

5. References

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