Introduction to Financial Derivatives and Option Pricing

Technical Report by Norbert Fogarasi (Jan 2003)

1.INTRODUCTION

The purpose of this document is to define the basic terms used in financial option pricing theory and attempt to briefly summarize the work done in the area to date. It is intended to stimulate interest in the topic and also to state the directions of research, which I am interested in.

2.DEFINITIONS

Financial Derivatives have become increasingly important in the world of finance over the past 25 years. Today, a large percentage of global financial trading is made up by the exchange of such instruments. A *derivative* can be defined as a financial product, whose value depends on (derived from) the values of other, underlying variables. In most cases, the underlying variable is the price of another financial instrument (the *underlying instrument*). For example, a call option on Yahoo stock is a financial derivative, whose value depends on the price of Yahoo shares on the stock market.

Derivatives can be divided into two large categories: ones, where the owner of the derivative is OBLIGED to enter into a future transaction based on the underlying variable, and those, where the owner of the derivative has the choice or OPTION to enter into a future transaction, again based on the underlying variable.

The most popular of the first class is the so-called *forward contracts*. This is an agreement to buy or sell a given asset at a certain future time for a certain price. *Forward rates* and *forward prices* are usually established in such a way, that there is no initial payment by either party at the time of entering into the transaction. For example, a Hungarian company, which knows that it will need to make a payment of 1 million GBP (Great Britain Pounds) in 6 months time, but holds all of its assets in HUF (Hungarian Forint) can eliminate its risk stemming from the fluctuation of the GBP/HUF foreign exchange rate by entering into a 6-month *FX* (foreign exchange) forward contract. Assuming that the forward rate agreed with the counterparty of the forward contract today is 375 HUF/GBP, the company will be able to buy the 1 million GBP in 6 months' time for 375 million HUF, regardless of the actual exchange rate at the time. This agreement can be quite profitable to the company, as it "*hedges* the FX risk"; if the actual exchange rate in 6 months time is only 400 HUF/GBP, the company will be able to obtain the 1 million GBP for 375 million HUF according to the terms of the contract, despite the fact that in the current market, it would cost 400 million HUF. On the other hand, the forward contract is an obligation, so if the actual exchange rate is 350 HUF/GBP in 6 months' time, the company will still have to fork out 375 million HUF, thus entering the forward contract actually resulted in a loss of 25 million HUF.

As the previous example illustrates, the forward contract protects well against unfavorable market movements, but it does not allow the holders to participate in gains, in case the market moves in their favor. For this reason, *financial options* have become more popular risk hedging instruments to corporations than forward contracts. In the above example, the Hungarian firm could enter into a 6-month *call option* on the HUF/GBP FX rate. The *call option* gives the holder the right, but not an obligation, to buy a certain underlying asset at a future date at a pre-agreed price (so-called *exercise price* or *strike price*). In our example, the company can purchase the option to buy 1 million GBP for 375 million HUF in 6 months' time. If the actual exchange rate is 400 HUF/GBP, the company will exercise its option, and realize a gain of 25 million HUF, but if the exchange rate happens to drop to 350 HUF/GBP, the company can simply "let the option *expire*" and purchase the 1 million GBP at the market for 350 million HUF. The opposite of a call option is a *put option*, whose *holder* has the right, but not the obligation to sell a given asset for pre-agreed strike price. For example, a speculative investor who believes that the price of MATAV shares will decrease on the Budapest Stock Exchange, may buy a put option. If indeed the price of the shares decreases below the exercise price, the investor will be able to realize a profit by exercising the put option.

The financial options described in the previous paragraph had one fixed exercise date in the future, these are referred to as *European options*. Options, which can be exercised any time between the start date and a future date are referred to as *American options*. For example, if the Hungarian company is not certain about the exact timing of its future GBP cashflow, it may purchase and American FX Option to hedge the exposure. Options, which can be exercised only on certain dates between the start date and the final exercise date are referred to as *Bermudan options*. Even more complex are *Asian options*, where the final value depends on the average price of the underlying asset, during at least some part of the life of the option. Asian options are an example of *path-dependent options*, where the final value of the option depends on the path followed by underlying variable, not just the final value of it.

The buyer or holder of any derivative is said to have a *long position* in the derivative, while the seller a *short position*.

A good way to characterize financial derivatives is by their *payoff function*: how much is the derivative worth at the time of expiration, as a function of the value of the underlying variable at maturity. The payoff function for owning a forward contract to buy/sell and call/put options is depicted in the below diagrams.

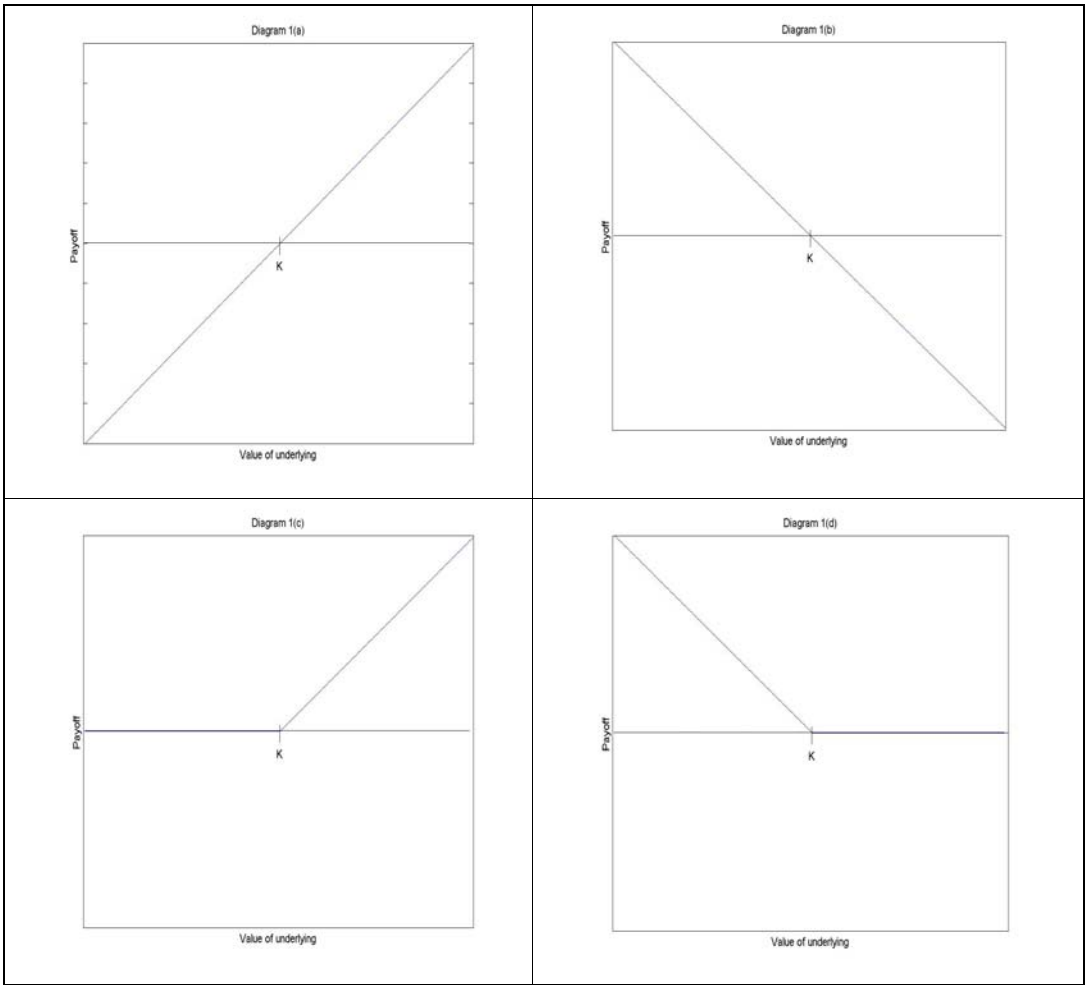
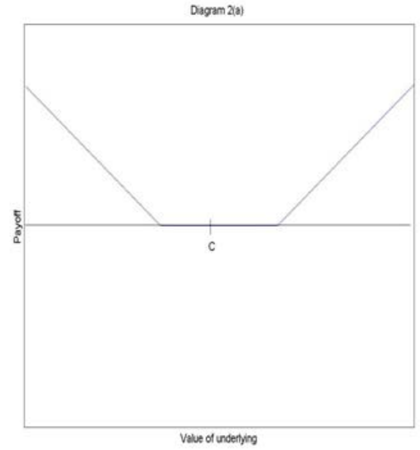


Diagram 1. The payoff function of various basic derivatives:

(a) long position in a forward contract, K=forward price
(b) short position in a forward contract, K=forward price
(c) long position in a European call option, K=strike price
(d) long position in a European put option, K=strike price

Indeed, it is not difficult to see that by buying and selling call and put options struck at various exercises prices, we can construct arbitrarily complicated payoff functions - two examples, which correspond to popular trading positions are explained below.



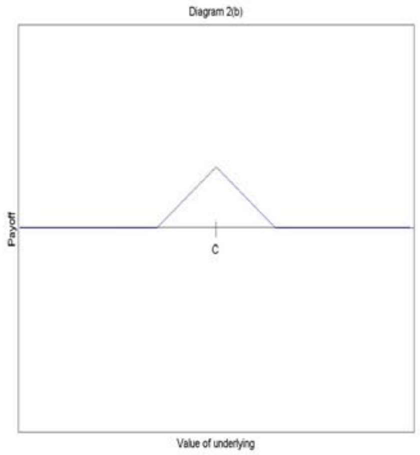


Diagram 2. Two examples of payoff functions for more complex derivative positions

(a) Strangle position: long positions in a put and a call with different strike prices. As long as the value of underlying moves greatly from the current price of underlying (C), the strangle position has a positive payoff. A speculative investor who believes that the underlying price will differ greatly from the current price may buy a strangle.
(b) Butterfly spread position: A combination of 4 different call/put options struck at 3 different strike prices. The holder of this position will benefit most, if the future price of the underlying does not move at all from the current price (C). A speculative investor, who predicts little movement in the future price of the underlying may buy a butterfly spread.

Derivatives have been introduced for a vast array of underlying financial instruments. The main areas are summarized below:

* *Equity derivatives*: The underlying variable is the price of a stock, or collection of stocks or the level of an index. Eg. option on Yahoo shares, option on BUX index
* *Fixed income derivatives*: The underlying variable is the future level of interest rate. Eg. interest rate cap (an option that provides a payoff when a specified interest rate is above a certain level), interest rate floor (opposite)
* *Commodity derivatives*: The underlying variable is the price of some physical commodity. Eg. futures on corn, oil or hogs.
* *FX derivatives*: The underlying variable is the foreign exchange rate between two currencies. Eg. FX forward contract, option on the USD/HUF exchange rate

3.HOW MUCH IS AN OPTION WORTH?

By definition, at the time of exercise, the option is worth either a positive amount (in this case we say the option is *in the money*) or zero (the option is *out of the money*). Because it is a right, not an obligation to enter into a future transaction, under no circumstance can it cause a loss to its holder, therefore an option always has non-negative value.

There are some other basic relationships, which can be observed from the above definitions, between the variables used to define an option, and its value (eg. higher current underlying price implies a higher value for a European call). Furthermore, some very loose upper and lower bounds can be derived for European and American call and put prices, without making any assumptions about the underlying process; however, these are of very limited practical value. After a cursory study of the option pricing problem, it becomes clear that to derive any practically usable results, one has to make assumptions about the underlying process – in particular, about the future probability distribution for it.

For sake of simplicity, we will consider the problem of pricing a European option on a non-dividend-paying stock. A good introductory study of the problem is presented by making the assumption that the underlying follows a *one-step binomial process* (ie. at the time of option maturity, it can only have two possible values: one is in the money, the other out of the money). Cox, Ross and Rubinstein in 1979 showed that in this simple case, it is possible to construct a portfolio of underlying stocks and options which has the same value regardless of which of the two possible values the underlying stock takes on at the time of maturity (we call this a *riskless portfolio*) This construction allowed them to deduce a price for the option in terms of the initial stock price. More precisely, they showed that if the price were anything else, there would exist a portfolio, which would allow the holder to realize a riskless profit, or *arbitrage*. This method of option pricing is called *arbitrage-free valuation*, and is an important way to deduce results. While it is clear that the one-step binomial model makes too strong an assumption about the underlying process to be practically useful, its extension to *multi-step binomial trees* provides a good framework for studying discrete models of option pricing. These provide practically useful numerical techniques for valuing both European and American options, and are used in current market practice to value exotic options with complex payoffs or early-exercise features.

A natural extension of the above work is to consider continuous models for the future behavior of the underlying stock price. In particular, because future price movements are uncertain, we use *continuous-time stochastic processes* for modeling. The most widely studied case is the assumption that stock prices follow a *geometric Brownian motion* with constant drift (μ) and volatility (σ), which implies that the future probability distribution of the stock price is *lognormal*. The most important result of option-pricing theory was achieved in 1973 by Fischer Black, Myron Scholes and Robert Merton, who used arbitrage-free valuation to show that under this continuous model, there is a closed-form formula for the price of European call and put options. For this important result, known as the Black-Scholes (BS) formula and widely used in current market practice, Robert Merton and Myron Scholes were awarded the Nobel prize for economics in 1997.

1. The underlying process is a non-dividend-paying stock. There are no transaction costs or taxes, securities are perfectly divisible and short-sellable and trading is continuous.
2. The risk-free interest rate is constant and the same for all maturities.
3. The underlying process follows a geometric Brownian motion with constant drift and volatility
4. There are no riskless arbitrage opportunities (market is efficient)

Diagram 3. The list of assumptions made by the Black-Scholes option pricing model

4. EXTENSIONS TO THE BLACK-SCHOLES MODEL AND RESEARCH DIRECTIONS

There are two distinct directions in post-BS options pricing research: European option pricing under looser assumptions than those made in the BS model, and valuation models for more complex and non-European options.

In the first category, there are important practical results relating to removing the non-dividend-paying assumption, removing the assumption regarding no transaction costs or taxes, introducing some limitations on liquidity (*bid-offer spread*) and most recently, incorporating risks due to the credit-worthiness of the counterparty. However, the most fundamental area of research concerns the very heart of the BS analysis: the assumption that stocks follow a geometric Brownian motion, and hence possess a lognormal probability distribution. There is strong empirical evidence to suggest that, for example, daily changes in equity prices and Foreign Exchange rates are not lognormally distributed, as the BS assumption would imply, but have heavier tails (Hull-White 1998, Hull 2002) In order to compensate for this effect, in practice, traders adjust the volatility input parameter to the Black-Scholes formula, based on the strike price and the time to maturity of the option (this is known as the *volatility smile*). These practical adjustments are heuristic, and an active area of research is to find a scientific model, which would encompass these considerations in a systematic way (attempts include the CEV model, jump diffusion model, stochastic volatility model, IVF model).

The second category contains development of numerical procedures to price more complex financial derivatives, still under the BS assumptions. The major results show that numerical methods, which come from extending the above-described binary tree idea (eg. general tree-pricing methods, *finite-difference methods* and *adaptive meshes*) are well suited to valuing American-style options. Another very active area of research is the use of *Monte-Carlo simulation* methods, which have been shown to work well for path-dependent options and for options where there are multiple underlying stochastic variables. Interesting open questions include whether Monte Carlo simulation method can be used for American option pricing, what extensions to these numerical methods are practically useful for pricing exotic options and whether there exist closed-form formulas for approximating the prices of exotic options well enough to be practically useful (although the recent advancements in computer hardware and increased sophistication of numerically-based option pricing systems have removed the practical interest in this latter area in recent years).

5. RESEARCH DIRECTIONS OF MY INTEREST

Of the above described research directions, I am most interested in exploring non-Gaussian models of option pricing. In particular, I am interested in developing models for the underlying process where we relax the Brownian assumption, thereby extending the probability distribution of the future value of the underlying to be a more generic family of functions (eg. exponential). Furthermore, I am interested in how we can then use historical data to calibrate such models in an adaptive way. Intuitively, older data matters less than recent data in estimating the future probability distribution, so this problem lends itself well to recently developed adaptive or on-line fitting techniques. I am interested in implementing these models and calibration techniques and examining their viability with real-life underlying and option data in a variety of markets (FX, equity and fixed income).

6. REFERENCES

Black, F. and M. Scholes. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81:637-59, 1973.

Cox, JS. Ross and M. Rubinstein. Option Pricing: A Simplified Approach. *Journal of Financial Economics*. 7:229-64, 1979

J.C. Hull. *Options, Futures and Other Derivative Securities.* Prentice-Hall, Englewood Cliffs, New Jersey, 5th edition, 2003.

J.C. Hull and A. White. An Analysis of the Bias in Option Pricing Caused by a Stochastic Volatility. *Advances in Futures and Options Research.* 3:27-61, 1998.

J.C. Hull and W. Suo. A Methodology for the Assessment of Model Risk and its Application to the Implied Volatility Function Model. *Journal of Financial and Quantitative Analysis.* 37/2, 2002.