SAMPLING FORMULAS

We briefly review two alternative ways of understanding the basic sampling formulas which are at the heart of Shannon's theory. To simplify the argument, we use a normalized time scale with a sampling step $T = 1$.

A. Sampling and Dirac Distributions

It is a common engineering practice to model the sampling process by a multiplication with a sampling sequence of Dirac impulses

$$\sum_{k\in Z} \delta(x-k) \overset{\text{Fourier}}{\longleftrightarrow} 2\pi \sum_{m\in Z} \delta(\omega + 2\pi m). \quad \text{(A.1)}$$

The corresponding sampled signal representation is

$$fs(x) = \sum_{k\in Z} f(k)\delta(x-k). \quad \text{(A.2)}$$

In the Fourier domain, multiplication corresponds to a convolution, which yields

$$\hat{f}_\delta(\omega) = \hat{f}(\omega) * \sum_{m\in Z} \delta(\omega + 2\pi m) = \sum_{m\in Z} \hat{f}(\omega + 2\pi m) \quad \text{(A.3)}$$

where the underlying Fourier transforms are to be taken in the sense of distributions. Thus, the sampling process results in a periodization of the Fourier transform of f, as illustrated in Fig. 1(b). The reconstruction is achieved by convolving the sampled signal $f_\delta(x)$ with the reconstruction function φ

$$f_{\text{rec}}(x) = (f_\delta * \varphi)(x) = \sum_{k\in Z} f(k)\varphi(x-k). \quad \text{(A.4)}$$

In the Fourier transform domain, this gives

$$\hat{f}_{\text{rec}}(\omega) = \hat{\varphi}(\omega) \cdot \sum_{m\in Z} \hat{f}(\omega + 2\pi m). \quad \text{(A.5)}$$

Thus, as illustrated in Fig. 1(c), we see that a perfect reconstruction is possible if $\hat{\varphi}(\omega)$ is an ideal low-pass filter [e.g., $\varphi(x) = \text{sinc}(x)$] and $\hat{f}(\omega) = 0$ for $|\omega| > \pi$ (Nyquist criterion).

If, on the other hand, f is not bandlimited, then the periodization of its Fourier transform in (A.5) results in spectral overlap that remains after postfiltering with $\hat{\varphi}(\omega)$. This distortion, which is generally nonrecoverable, is called *aliasing*.

Theorem 1 [Shannon]: If a function $f(x)$ contains no frequencies higher than $\omega_{\max}$ (in radians per second), it is completely determined by giving its ordinates at a series of points spaced $T = \pi/\omega_{\max}$ seconds apart.

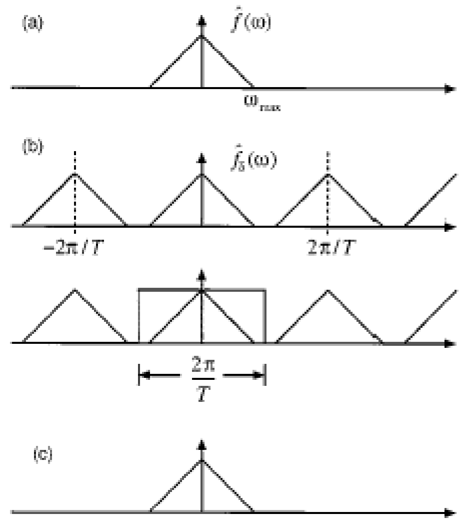

Fig. 1. Frequency interpretation of the sampling theorem: (a) Fourier transform of the analog input signal $f(x)$, (b) the sampling process results in a periodization of the Fourier transform, and (c) the analog signal is reconstructed by ideal low-pass filtering; a perfect recovery is possible provided that $\omega_{\max} \leq \pi/T$.

The reconstruction formula that complements the sampling theorem is

$$f(x) = \sum_{k\in Z} f(kT)\,\text{sinc}(x/T - k) \quad (1)$$

in which the equidistant samples of $f(x)$ may be interpreted as coefficients of some basis functions obtained by appropriate shifting and rescaling of the sinc-function: $\text{sinc}(x) = \sin(\pi x)/(\pi x)$. Formula (1) is exact if $f(x)$ is bandlimited to $\omega_{\max} \leq \pi/T$; this upper limit is the Nyquist frequency, a term that was coined by Shannon in recognition of Nyquist's important contributions in communication theory [88].

B. Sampling and Poisson's Summation Formula

The standard form of Poisson's summation formula is (see [117])

$$\sum_{k \in Z} g(k) = \sum_{m \in Z} \hat{g}(2\pi m) \tag{A.6}$$

where $\hat{g}(\omega)$ denotes the Fourier transform of the continuous time function $g(x) \in L_1$. The reader is referred to [13], [16], or [65] for a rigorous mathematical treatment.

Considering the function $g(x) = f(x)e^{-\omega_0 x}$, the Fourier transform of which is $\hat{g}(\omega) = \hat{f}(\omega + \omega_0)$ (modulation property), we get

$$\sum_{k \in Z} f(k)\, e^{-j\omega_0 k} = \sum_{m \in Z} \hat{f}(\omega_0 + 2\pi m) = F(e^{j\omega_0}). \tag{A.7}$$

This is precisely the discrete-time Fourier transform of the sequence $\{f(k)\}$ with ω_0 as the frequency variable. The central term of (A.7) is identical to (A.3), which means that the 2π-periodic functions $F(e^{j\omega})$ and $\hat{f}_\delta(\omega)$ are in fact equivalent, even though they have very different interpretations—the former is a discrete-time Fourier transform, while the latter is a continuous-time one.

The last step in this formulation is to derive the Fourier transform of the reconstructed signal

$$\hat{f}_{\text{rec}}(\omega) = \int_{-\infty}^{+\infty} \left(\sum_{k \in Z} f(k)\varphi(x-k) \right) e^{-j\omega x}\, dx.$$

Exchanging the order of integration and making the change of variable $y = x - k$, we get

$$\begin{aligned} \hat{f}_{\text{rec}}(\omega) &= \sum_{k \in Z} f(k)\, e^{-j\omega k} \int_{-\infty}^{+\infty} \varphi(y)\, e^{-j\omega y}\, dy \\ &= F(e^{j\omega}) \cdot \hat{\varphi}(\omega). \end{aligned} \tag{A.8}$$

Together with (A.7), (A.8) is equivalent to (A.5).

Sampling—50 Years After Shannon

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