SAMPLING CRITERION FOR NONLINEAR SYSTEMS
WITH A BANDPASS INPUT

Ching-Hsiang Tseng

Department of Electrical Engineering
National Taiwan Ocean University
2 Pei-ning Rd., Keelung, Taiwan, R.O.C.

ABSTRACT
Sampling requirements for nonlinear systems with a bandpass input are developed in this paper. It is well known that the output of a nonlinear system may have a larger bandwidth than that of the input. According to the Nyquist sampling theorem, the sampling rate needs to be at least twice the maximum frequency of the output to avoid aliasing. However, if the input is a bandpass signal, the spectrum of the output is usually distributed over several frequency bands. In this case, using the bandpass sampling concept, it is possible to sample the output at a much lower rate. In this paper, conditions for such a lower sampling rate to exist are derived for nonlinear systems up to the third order. Supporting computer simulations are also provided.

1. INTRODUCTION
Nonlinear systems often have a spectral spreading effect on their input signals. For an N-th order nonlinear system, the maximum frequency of the output signal can be as large as N times the maximum frequency of the input signal. In general, the output signal of the nonlinear system needs to be sampled at a rate which is at least 2N times the maximum frequency of the system input, otherwise the output samples would be aliased [1]. It has been shown that, for identification of nonlinear systems, it suffices for the sampling rate of the output signal to be twice the maximum frequency of the input signal, even though the resulting output samples are aliased [2, 3]. However, for situations where retaining all information in the output signal is the main concern and system identification is of little interest, avoiding aliasing in the output samples to ensure proper reconstruction of the original output signal is desired.

In many cases, such as digital communications over nonlinear channels [4], etc., the input to the nonlinear system is a bandpass signal whose spectrum is as shown in Fig. 1(a). The bandpass signal has no spectral components below \( f_L \) Hz or above \( f_H \) Hz, and has a bandwidth of \( B = f_H - f_L \). The bandpass sampling theorem [5] states that such a bandpass signal can be sampled at a frequency \( f_s \) without causing aliasing if \( f_s \) satisfies

\[
\frac{2QB}{n} \leq f_s \leq \frac{2(Q-1)B}{n-1},
\]

where \( Q = f_H/B \), and \( n \) is a positive integer and \( n \leq Q \). For \( n = 1 \), eq. (1) simply states the Nyquist sampling criterion. For \( n > 1 \), \( f_s \) is smaller than the Nyquist sampling frequency. The larger the value of \( n \), the smaller the sampling frequency. Note that the output of the nonlinear system to such a bandpass signal may possess multiple frequency bands. We see that (1) can not be applied to the output signal.

It is the objective of this paper to derive the sampling requirements for the output signal of nonlinear systems excited by a bandpass input signal. Considering that the cubically nonlinear system is the lowest-order nonlinear system including both even and odd order nonlinear terms, we derive the bandpass sampling requirements for nonlinear systems up to the third order. In fact, one may find in the literature that many nonlinear effects in science and engineering can be appropriately modeled as a cubically nonlinear system. Extension of the derivation to higher-order nonlinear systems, although may be quite involved, is conceptually straightforward.

2. BANDPASS SAMPLING FOR QUADRATICALLY NONLINEAR SYSTEMS
Consider a quadratically nonlinear system (including linear and quadratic responses) with an input \( x(t) \) and an output \( y(t) \). Suppose that \( x(t) \) is a bandpass signal whose spectrum, say \( X(f) \), is shown in Fig. 1(a), then the spectrum of the output \( y(t) \), say \( Y(f) \), would have a form like Fig. 1(b). Note that, in Fig. 1(b), the spectrum labeled '1' is owing to the linear response of the system, and the spectra labeled '0' and '2' are owing to the quadratic response of the system.
If the output is sampled at \( f_s \) Hz, then the spectrum of the sampled output can be obtained by replicating the spectrum of the original signal at multiples of \( f_s \). To be immune from aliasing, neither spectrum ‘1’ nor spectrum ‘2’ can straddle \( n f_s \) or \((n + 1/2)f_s\) for any integer \( n \). Because if a spectrum (say spectrum ‘1’) straddles \( n f_s \), then its counterpart in the negative frequency region (labeled as spectrum ‘-1’) must straddle \(-n f_s\). Due to spectral replication, the spectrum of the sampled output would have the positive and the negative frequency spectra overlapping in the vicinity of every multiple of \( f_s \). This leads to aliasing in the sampled output. Similar result can be reached for the case of a spectrum straddling \((n + 1/2)f_s\).

Let the center frequency of spectrum ‘i’ be \( f_i \), \( i = \pm 1, \pm 2 \). Note that \( f_2 = 2f_1 \). We must have \( n f_s < f_1 < (n + 1)f_s \) (and thus \( 2nf_s < f_2 < 2(n + 1)f_s \)) for some \( n \). Referring to the frequency band \([kf_s, (k + 1)f_s]\) (\( k \) an integer) as a segment \( k \), we then define \( \epsilon_i \) as the relative position of \( f_i \) in its corresponding segment. For example, \( f_1 \) is in segment \( n \), therefore \( \epsilon_1 = f_1 - nf_s \). The center frequency \( f_2 \) of \( f_2 \) can be either in segment \( 2n \) or segment \( 2n + 1 \), therefore, \( \epsilon_2 = R_{f_2} f_1 \), \((f_2 - 2nf_s) = R_{f_2} f_1(2\epsilon_1)\), where \( R_{f_2} \) denotes the remainder of \( \frac{b}{2} \) divided by \( a \).

Due to spectral replication, a replica of spectrum ‘i’ (\( i = \pm 1, \pm 2 \)) will appear in each segment. It is easy to see that the positions of the replicas ‘1’ and ‘-1’ will be symmetric with respect to the center frequency \( f_s/2 \) in each segment, so will the positions of the replicas ‘2’ and ‘-2’. Note that \( \epsilon_1 \) is in fact the relative position of the center frequency of the replica ‘i’ in each segment. To be immune from aliasing, the replicas in each segment can not overlap. To derive the sampling requirements, it is sufficient to consider the range \( 0 < \epsilon_1 < f_s \), which is further divided into the following six subranges:

**Case 1:** \( 0 < \epsilon_1 < \frac{1}{4} f_s \), and hence \( 0 < \epsilon_2 < \frac{3}{8} f_s \)

Recall that ‘1’ and ‘-1’ (as well as ‘2’ and ‘-2’) are symmetric with respect to \( f_s/2 \) in each segment, we have \( \frac{3}{4} f_s < \epsilon_1 < \frac{1}{4} f_s \) and \( \frac{3}{4} f_s < \epsilon_2 < \frac{1}{4} f_s \). Since \( \epsilon_1 < 2\epsilon_1 = \epsilon_2 \), the spectrum of the sampled output would have a form like Fig. 2. In Fig. 2, we use solid circles to denote the spectra of the original output signal, and use dashed circles to denote the spectra of the replicas. The number in each circle indicates where in Fig. 1(b) it is from. For example, a dashed circle labeled ‘-2’ denotes a replica of the spectrum ‘-2’.

From Fig. 2 we see that the requirements for no spectral overlapping are

\[
\begin{align*}
\frac{f_L}{f_s} & \geq n f_s + B \\
\frac{2f_L}{f_s} & \geq f_H + n f_s \\
\frac{3f_L}{f_s} & \geq (2n + \frac{1}{2})f_s
\end{align*}
\]

which leads to

\[
\frac{3QB}{4n + 1} \leq \frac{f_s}{2} \leq \frac{(Q - 2)B}{n}
\]

**Case 2:** \( \frac{1}{4} f_s < \epsilon_1 < \frac{3}{8} f_s \), and hence \( \frac{3}{8} f_s < \epsilon_2 < \frac{3}{8} f_s \)

In this case, we have \( \frac{3}{4} f_s < \epsilon_1 < \frac{3}{4} f_s \) and \( \frac{3}{4} f_s < \epsilon_2 < \frac{3}{4} f_s \). The resulting spectrum of the sampled output is shown in Fig. 3. We see from Fig. 3 that the requirements for no spectral overlapping are

\[
\begin{align*}
\frac{f_L}{f_s} & \geq n f_s + B \\
\frac{3f_L}{f_s} & \geq (2n + \frac{1}{2})f_s
\end{align*}
\]

which yields

\[
\frac{3QB}{3n + 1} \leq \frac{f_s}{2} \leq \frac{4(Q - 1)B}{4n + 1}
\]
Case 3: \( \frac{1}{3} f_s < \epsilon_1 < \frac{1}{2} f_s \), and hence \( \frac{3}{2} f_s < \epsilon_2 < f_s \).

In this case, we have \( \frac{1}{2} f_s < \epsilon_1 < \frac{3}{2} f_s \) and \( 0 < \epsilon_2 < \frac{3}{2} f_s \). Therefore, the spectrum of the resulting sampled output should be like that shown in Fig. 4. The requirements for preventing spectral overlapping in this case are as follows:

\[
2 f_L \leq (2n + 1) f_s - B \tag{10}
\]

\[
f_H \geq n f_s + [(2n + 1) f_s - 2 f_L] \tag{11}
\]

\[
f_H \leq (n + \frac{1}{2}) f_s \tag{12}
\]

This results in

\[
\frac{(2Q + 1)B}{2n + 1} \leq f_s \leq \frac{3(Q - 1)B}{3n + 1} \tag{13}
\]

Case 4: \( \frac{1}{3} f_s < \epsilon_1 < \frac{3}{4} f_s \), and hence \( 0 < \epsilon_2 < \frac{1}{2} f_s \).

In this case, we have \( \frac{1}{3} f_s < \epsilon_1 < \frac{3}{4} f_s \) and \( 0 < \epsilon_2 < \frac{1}{2} f_s \). Note that since \( f_s < 2 \epsilon_1 < \frac{3}{4} f_s \), the spectrum '2' is in segment \( 2n + 1 \) instead of \( 2n \). Therefore, the spectrum of the sampled output for this case is shown in Fig. 5. To avoid spectral overlapping, we must have

\[
2 f_L \geq (2n + 1) f_s + B \tag{14}
\]

\[
f_H \leq (n + 1) f_s - [2 f_H - (2n + 1) f_s] \tag{15}
\]

\[
f_L \geq (n + \frac{1}{2}) f_s \tag{16}
\]

which gives the following

\[
\frac{3Q B}{3n + 2} \leq f_s \leq \frac{(2Q - 3)B}{2n + 1} \tag{17}
\]

Case 5: \( \frac{3}{4} f_s < \epsilon_1 < \frac{3}{4} f_s \), and hence \( \frac{3}{4} f_s < \epsilon_2 < \frac{3}{2} f_s \).

In this case, we have \( \frac{1}{3} f_s < \epsilon_1 < \frac{1}{2} f_s \) and \( \frac{1}{3} f_s < \epsilon_2 < \frac{3}{2} f_s \). Since \( \frac{3}{4} f_s < 2 \epsilon_1 < 2 f_s \), the spectrum '2' is in segment \( 2n + 1 \). The spectrum of the sampled output for this case is shown in Fig. 6. The requirements for no spectral overlapping in Fig. 6 are

\[
f_H \leq (n + 1) f_s - B \tag{18}
\]

\[
2 f_L \geq (2n + 1) f_s + [(n + 1) f_s - f_L] \tag{19}
\]

\[
2 f_H \leq (2n + \frac{3}{2}) f_s \tag{20}
\]

which leads to

\[
\frac{4QB}{4n + 3} \leq f_s \leq \frac{3(Q - 1)B}{3n + 2} \tag{21}
\]

Case 6: \( \frac{3}{4} f_s < \epsilon_1 < f_s \), and hence \( \frac{1}{2} f_s < \epsilon_2 < f_s \).

In this case, we have \( 0 < \epsilon_1 < \frac{1}{4} f_s \) and \( 0 < \epsilon_2 < \frac{1}{2} f_s \). Since \( \frac{3}{4} f_s < 2 \epsilon_1 < 2 f_s \), the spectrum '2' is again in segment \( 2n + 1 \). It is easy to see that, in this case, \( \epsilon_2 = R f_s \{2 \epsilon_1 \} \) must be smaller than \( \epsilon_1 \), hence the spectrum of the sampled output is as that shown in Fig. 7.

To avoid spectral overlapping in Fig. 7, we must have

\[
f_H \leq (n + 1) f_s - B \tag{22}
\]

\[
2 f_L \leq f_L + (n + 1) f_s \tag{23}
\]

\[
2 f_H \geq (2n + \frac{3}{2}) f_s \tag{24}
\]

which results in

\[
\frac{(Q + 1)B}{n + 1} \leq f_s \leq \frac{4(Q - 1)B}{4n + 3} \tag{25}
\]

3. BANDPASS SAMPLING FOR CUBICALLY NONLINEAR SYSTEMS

Consider a cubically nonlinear system (which includes linear, quadratic, and cubic responses). Suppose the input to
4. SIMULATION

A bandpass signal \( f_L = 990 \text{ Hz}, f_H = 1000 \text{ Hz}, \) and \( B = 10 \text{ Hz} \) is used as the input to the cubically nonlinear system \( y(t) = x(t) + x^2(t) + x^3(t) \). The spectrum of the system output is shown in Fig. 9(a), where 4 frequency bands at around 0, 1000, 2000, and 3000 Hz are clearly shown. Using \( n = 3 \), we found 273.64 < \( f_s \) < 275 is the valid range for the case of spectral order \( \{-3, -2, -1, 1, -2, 3\} \). We chose \( f_s = 274 \text{ Hz} \) to sample the output signal. The spectrum of the sampled output is shown in Fig. 9(b). The result indicates that various spectra do not overlap, and the spectral order is indeed \( \{-3, -2, -1, 1, -2, 3\} \).

Fig. 8. The spectrum of the output signal of a cubically nonlinear system.

Fig. 9. Spectra of (a) the original and (b) the sampled output signals in the simulation.

5. CONCLUSION

Bandpass sampling requirements for nonlinear systems up to the third order were derived. The results can be used to select appropriate sampling frequency for the output of nonlinear systems with a bandpass input.

6. REFERENCES


