Jules Lissajous and His Figures

Jules Antoine Lissajous (1822–1880) is not among the giants in the history of science, yet his name is known to physics students through the "Lissajous figures"—patterns formed when two vibrations along perpendicular lines are superimposed. Lissajous entered the École Normale Supérieure in 1841 and later became professor of physics at the Lycée Saint-Louis in Paris, where he studied vibrations and sound. In 1855 he devised a simple optical method for studying compound vibrations: he attached a small mirror to each of the vibrating objects (two tuning forks, for example) and aimed a beam of light at one of the mirrors. The beam was reflected first to the other mirror and thence to a large screen, where it formed a two-dimensional pattern, the visual result of combining the two vibrations. This simple idea—a forerunner of the modern oscilloscope—was a novelty in Lissajous' time, for up until then the study of sound depended solely on the process of hearing, that is, on the human ear. Lissajous literally made it possible to "see sound."

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Let each vibration be a simple harmonic motion represented by a sinusoidal wave; let a and b denote the amplitudes, ω_1 and ω_2 the angular frequencies (in radians per second), ϕ_1 and ϕ_2 the phases, and t the time. We then have

$$x = a\sin(\omega_1 t + \phi_1), \ y = b\sin(\omega_2 t + \phi_2). \qquad (1)$$

As time progresses, the point P whose coordinates are (x, y) will trace a curve whose equation can be found by eliminating t between equations (1). Since the two equations contain six parameters,[1] the curve is usually quite complicated, except in some special cases. For example, if $\omega_1 = \omega_2$ and $\phi_1 = \phi_1$, we have

$$x = a\sin(\omega t + \phi), \ y = b\sin(\omega t + \phi),$$

where we have dropped the subscripts under the parameters. To eliminate t between these equations, we note that $x/a = y/b$ and thus $y = (b/a)x$, the equation of a straight line. Similarly, for $\omega_1 = \omega_2$ and a phase difference of π we get the line $y =$

$-(b/a)x$. For $\omega_1 = \omega_2$ and a phase difference of $\pi/2$, we have (taking ϕ_1 to be zero)

$$x = a \sin \omega t, \, y = b \sin (\omega t + \pi/2) = b \cos \omega t.$$

Dividing the first equation by a and the second by b, squaring the results, and adding, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which represents an ellipse with axes along the x- and y-axes (if in addition $a = b$, the ellipse becomes a circle). For an arbitrary phase difference the curve will be a tilted ellipse, of which the previous examples are special cases (the lines $y = \pm(b/a)x$ being degenerate ellipses). If we let the phase difference vary continuously, the ellipse will slowly change its orientation and shape, passing (in the case $a = b$) from the circle $x^2 + y^2 = 1$ to the lines $y = \pm x$ and back (fig. 70).

   If the frequencies are unequal, the curve is much more complex. For example, when $\omega_2 = 2\omega_1$ (musically speaking, when the two vibrations are an octave apart), we have

$$x = a \sin \omega t, \, y = b \sin (2\omega t + \phi),$$

where we again dropped the subscripts and took $\phi_1 = 0$. The kind of curve we get now depends on ϕ. For $\phi = \pi/2$ we have

$$x = a \sin \omega t, \quad y = b \sin (2\omega t + \pi/2) = b \cos 2\omega t.$$

Using the identity $\cos 2u = 1 - 2 \sin^2 u$ and eliminating t between the two equations results in $y = b[1 - 2(x/a)^2]$. This equation represents a parabola along which the point P moves back and forth as time progresses. For other values of ϕ the curve may have a closed shape (fig. 71).

   One other observation is worth mentioning. As long as the frequency ratio ω_1/ω_2 is a rational number, the curve—no matter how complex—will eventually repeat itself, causing the motion to be periodic.[2] But if ω_1/ω_2 is irrational, the point P will never retrace its own path, resulting in a non-periodic motion. However, as time progresses the curve will gradually fill the rectangle bounded by the lines $x = \pm a$, $y = \pm b$ (fig. 72).

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Lissajous' work was praised by his contemporaries and discussed by the physicists John Tyndall (1820–1893) and Lord Rayleigh (John William Strutt, 1842–1919) in their classical treatises on acoustics. In 1873 he was awarded the prestigious Lacaze Prize
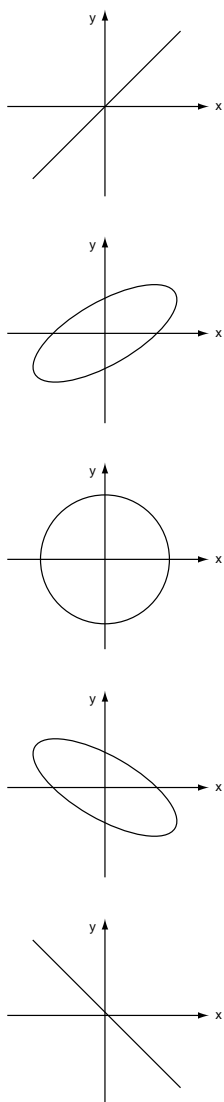
FIG. 70. Lissajous figures: the case $\omega_1 = \omega_2$.

for his "beautiful experiments," and his method was exhibited at the Paris Universal Exposition in 1867. But apparently nothing is new under the sun: Lissajous' figures had been discovered long before by the self-taught American scientist Nathaniel Bowditch (1773–1838), who produced them in 1815 with a compound pendulum.[3] A variation of this device, in which the motion of two pendulums is combined and traced on paper by means of a pen attached to one of the pendulums, became a popular nineteenth-century science demonstration (fig. 73); the
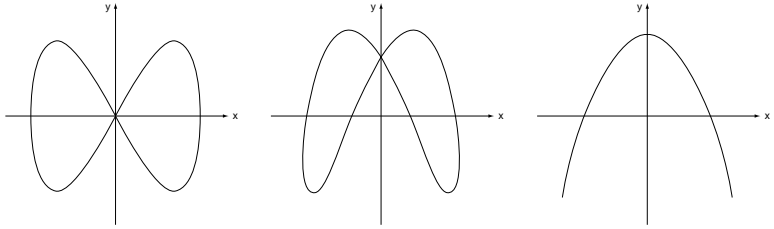
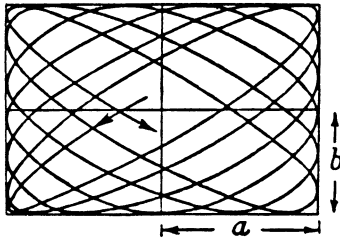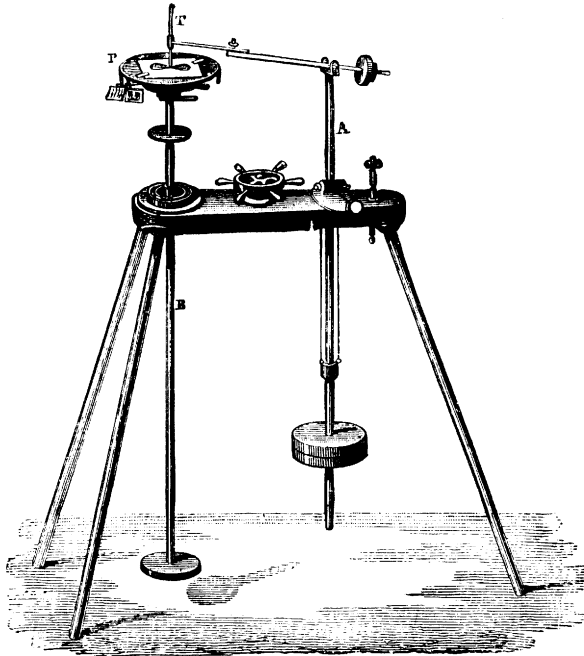**FIG. 71.** Lissajous figures: the case $\omega_2 = 2\omega_1$.

FIG. 72. Lissajous figure for the case when ω_1/ω_2 is irrational.

M. Tisley's Harmonograph.

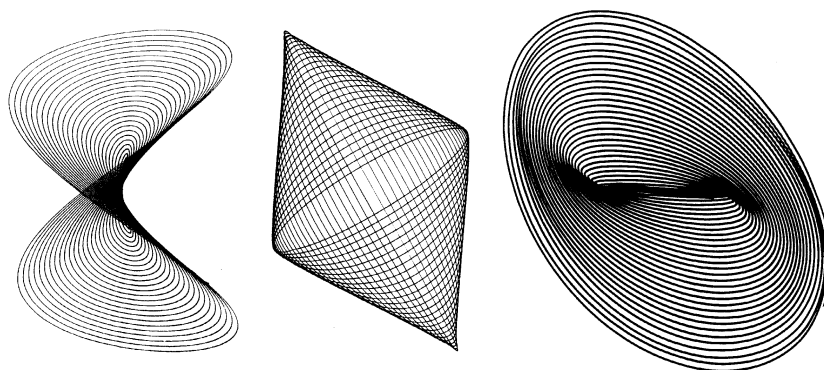**FIG. 73.** Harmonograph. From a nineteenth-century science book.

Fig. 74. Harmonograms.

ensuing figures were called "harmonograms," and their incredible variety never failed to impress the spectators (fig. 74).[4] The novelty of Lissajous' method was that it departed from mechanical devices and relied instead on the much more efficient agent of light. In this he was a visionary, foretelling our modern electronic era.

Notes and Sources

1. Actually five parameters are sufficient, since only the *relative* phase (i.e., the phase difference) matters.

2. The period will be the least common multiple of the two individual periods.

3. Florian Cajori, *A History of Physics* (1898, rev. ed. 1928; rpt. New York: Dover, 1962), pp. 288–289.

4. In 1980 my colleague Wilbur Hoppe and I built a compound pendulum as part of a workshop on mathematical models at the University of Wisconsin—Eau Claire. The patterns shown in figure 74 were produced with this device.

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