Risk analysis lab 2014. 11. 17. (Optimizing mean-reverting portfolios)

1. Load the supplied data from the binary (`s.mat`), the variable `s` will be a *TxN* matrix containing daily closing prices for *N* asset and *T* days ($s_t = \{s_1^{(t)}, s_2^{(t)}, \ldots, s_N^{(t)}\},\ \ t=1,\ldots,T$). For the given time series, calculate the covariance matrix **G**.

2. Optimizing mean-reverting portfolios

   a) Assume that s_t is subject to a first order vector autoregressive process – VAR(1) –, defined as follows: $s_t = A s_{t-1} + W_t$, where **A** is a matrix of size *NxN* and $W_t \sim N(0, \sigma I)$ are i.i.d.r.v.-s for some $\sigma > 0$.

   Estimate **A** using least squares estimation techniques, as $\hat{A} : \min_A \sum_{t=2}^{T} \| s_t - A s_{t-1} \|^2$,
   where $\|.\|^2$ denotes the Euclidian norm. Solving the minimization problem above, by equating the partial derivatives to zero with respect to each element of the matrix **A**, we obtain a system of linear equations. Solving that for **A** and switching back to vector notation for **s**, we obtain

   $$\hat{A} = \sum_{t=2}^{T} \left(s_{t-1}^T s_{t-1} \right)^+ \left(s_{t-1}^T s_t \right) \ ,$$

   where M^+ denotes the Moore-Penrose pseudoinverse of a matrix **M**. (Note that the Moore-Penrose pseudoinverse is preferred to regular matrix inversion, in order to avoid problems which may arise because of the potential singularity.)

   ```
   function A = est_A(s)
   ```

   b) The traditional way to identify the optimal sparse mean-reverting portfolio is to find a portfolio vector subject to maximizing its predictability. One may note that
   $w_{opt} : \max \lambda = \max_x \frac{w^T A G A^T w}{w^T G w}$ is equivalent to finding the eigenvector corresponding to the maximum eigenvalue in the following generalized eigenvalue problem: $A G A^T w = \lambda G w$. This can be transformed into a traditional eigenvalue problem by introducing the variable $u := G^{1/2} w,\ w = (G^+)^{1/2} u$ so that we have

   $$\left((G^+)^{1/2} \right)^T A G A^T (G^+)^{1/2} u = \lambda u \ .$$

   ```
   function w = mr_opt_w(A, G)
   ```

c) Estimating the long term mean (μ) of the process of portfolio valuations ($p_t = \boldsymbol{w}^T \boldsymbol{s}_t$) is instrumental for mean reverting trading. Rewriting the Ornstein-Uhlenbeck stochastic differential equation - $dp(t) = \theta(\mu - p(t))dt + \sigma\, dW(t)$ - to the following way: $s_t - s_{t-1} = \theta(\mu - s_{t-1})\Delta t + \sigma(W_t - W_{t-1}) = \theta\mu\Delta t - \theta s_{t-1}\Delta t + \sigma(W_t - W_{t-1})$ results in a linear regression in the form of $y = a + bx + \epsilon_t$, from which the estimation of the long term mean can be formulated as:

$$\hat{\mu} = \frac{a}{1-b} \quad .$$

(Hint: calculate the $\boldsymbol{p}$ vector then run a linear regression (`robustfit`) on the $\{p_t, p_{t-1}\}$ pairs.)

```
% pT :=  w^T s_T
function [mu, pT] = est_mu(s, w)
```